IGUSA'S p-ADIC LOCAL ZETA FUNCTION AND THE MONODROMY CONJECTURE FOR NON-DEGENERATE SURFACE SINGULARITIES

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Abstract. In 2011 Lemahieu and Van Proeyen proved the Monodromy Conjecture for the local topological zeta function of a non-degenerate surface singularity. We start from their work and obtain the same result for Igusa's p-adic and the motivic zeta function. In the p-adic case, this is, for a polynomial $f \in \mathbf{Z}[x, y, z]$ satisfying $f(0, 0, 0) = 0$ and non-degenerate with respect to its Newton polyhedron, we show that every pole of the local p-adic zeta function of f induces an eigenvalue of the local monodromy of f at some point of $f^{-1}(0) \subseteq \mathbf{C}^3$ close to the origin.

Essentially the entire paper is dedicated to proving that, for f as above, certain candidate poles of Igusa's p-adic zeta function of f, arising from so-called B_1-facets of the Newton polyhedron of f, are actually not poles. This turns out to be much harder than in the topological setting. The combinatorial proof is preceded by a study of the integral points in three-dimensional fundamental parallelepipeds. Together with the work of Lemahieu and Van Proeyen, this main result leads to the Monodromy Conjecture for the p-adic and motivic zeta function of a non-degenerate surface singularity.

Contents

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0. Introduction

0.1. **Igusa's zeta function and the Monodromy Conjecture.** For a prime p, we denote by $\mathbf{Q}_p$ the field of p-adic numbers and by $\mathbf{Z}_p$ its subring of p-adic integers. We denote by $|\cdot|$ the p-adic norm on $\mathbf{Q}_p$. Let $n \in \mathbf{Z}_{>0}$ and denote by $|dx| = |dx_1 \wedge \cdots \wedge dx_n|$ the Haar measure on $\mathbf{Q}_p^n$, so normalized that $\mathbf{Z}_p^n$ has measure one.

Definition 0.1 (Igusa's p-adic local zeta function)**.** Let p be a prime number, $f(x) = f(x_1, \ldots, x_n)$ a polynomial in $\mathbf{Q}_p[x_1, \ldots, x_n]$, and Φ a Schwartz–Bruhat function on $\mathbf{Q}_p^n$, i.e., a locally constant function $\Phi : \mathbf{Q}_p^n \to \mathbf{C}$ with compact support. Igusa's p-adic local zeta function associated to f and Φ is defined as

$$Z_{f,\Phi} : \{s \in \mathbf{C} \mid \Re(s) > 0\} \to \mathbf{C} : s \mapsto \int_{\mathbf{Q}_p^n} |f(x)|^s \Phi(x) |dx|.$$

We will mostly consider the case where Φ is the characteristic function of either $\mathbf{Z}_p^n$ or $p\mathbf{Z}_p^n = (p\mathbf{Z}_p)^n$. By Igusa's p-adic zeta function Z_f of f (without mentioning Φ), we mean $Z_{f,\Phi}$, where $\Phi = \chi(\mathbf{Z}_p^n)$ is the characteristic function of $\mathbf{Z}_p^n$. By the local Igusa zeta function Z_f^0 of f, we mean $Z_{f,\Phi}$, where $\Phi = \chi(p\mathbf{Z}_p^n)$ is the characteristic function of $p\mathbf{Z}_p^n$.

Using resolution of singularities, Igusa [26] proved in 1974 that $Z_{f,\Phi}$ is a rational function in the variable $t = p^{-s}$; more precisely, he shows that there exists a rational function $\widetilde{Z}_{f,\Phi} \in \mathbf{Q}(t)$, such that $Z_{f,\Phi}(s) = \widetilde{Z}_{f,\Phi}(p^{-s})$ for all $s \in \mathbf{C}$ with $\Re(s) > 0$. Denoting the meromorphic continuation of $Z_{f,\Phi}$ to the whole complex plane again with $Z_{f,\Phi}$, he also obtains a set of candidate poles for $Z_{f,\Phi}$ in terms of numerical data associated to an embedded resolution of singularities of the locus $f^{-1}(0) \subseteq \mathbf{Q}_p^n$. In 1984 Denef [11] proved the rationality of $Z_{f,\Phi}$ in an entirely different way, using p-adic cell decomposition.

For a prime number p and $f(x) = f(x_1, \ldots, x_n) \in \mathbf{Z}_p[x_1, \ldots, x_n]$, Igusa's zeta function Z_f is closely related to the numbers N_l of solutions in $(\mathbf{Z}_p/p^l\mathbf{Z}_p)^n$ of the polynomial congruences $f(x) \equiv 0 \bmod p^l$ for $l \geqslant 1$. For instance, the poles of Z_f determine the behavior of the numbers N_l for l big enough.

The poles of Igusa's zeta function are also the subject of the Monodromy Conjecture, formulated by Igusa in 1988. It predicts a remarkable connection between the poles of Z_f and the eigenvalues of the local monodromy of f. The conjecture is motivated by analogous results for Archimedean local zeta functions (over $\mathbf{R}$ or $\mathbf{C}$ instead of $\mathbf{Q}_p$) and—of course—by all known examples supporting it. If the Monodromy Conjecture were true, it would explain why generally only few of the candidate poles arising from an embedded resolution of singularities, are actually poles.

Conjecture 0.2 (Monodromy Conjecture for Igusa's p-adic zeta function over $\mathbf{Q}_p$)**.** [27]. *Let $f(x_1, \ldots, x_n)$ be a polynomial in $\mathbf{Z}[x_1, \ldots, x_n]$. For almost all*[1] *prime numbers p, we have the following. If s_0 is a pole of Igusa's local zeta function Z_f of f, then $e^{2\pi i \Re(s_0)}$ is an eigenvalue of the local monodromy operator acting on some cohomology group of the Milnor fiber of f at some point of the hypersurface $f^{-1}(0) \subseteq \mathbf{C}^n$.*

There is a local version of this conjecture considering Igusa's zeta function on a small enough neighborhood of $0 \in \mathbf{Q}_p^n$ and local monodromy only at points of $f^{-1}(0) \subseteq \mathbf{C}^n$ close to the origin.

Conjecture 0.3 (Local version of Conjecture 0.2)**.** *Let $f(x_1, \ldots, x_n)$ be a polynomial in $\mathbf{Z}[x_1, \ldots, x_n]$ with $f(0) = 0$. For almost all prime numbers p and for k big enough, we have the following. If s_0 is a pole of $Z_{f,\chi(p^k\mathbf{Z}_p^n)}$, then $e^{2\pi i \Re(s_0)}$ is an eigenvalue of the local monodromy of f at some point of the hypersurface $f^{-1}(0) \subseteq \mathbf{C}^n$ close to the origin.*

There exists a stronger, related conjecture, also due to Igusa and also inspired by the analogous theorem in the Archimedean case.

Conjecture 0.4. [27]. *Let $f(x_1, \ldots, x_n) \in \mathbf{Z}[x_1, \ldots, x_n]$. For almost all prime numbers p, we have the following. If s_0 is a pole of Z_f, then $\Re(s_0)$ is a root of the Bernstein–Sato polynomial $b_f(s)$ of f.*

[1]By 'almost all' we always mean 'all, except finitely many', unless expressly stated otherwise.

There is a local version of this conjecture considering $Z_{f,\chi(p^k\mathbf{Z}_p^n)}$ for k big enough and the local Bernstein–Sato polynomial $b_f^0(s)$ of f.

Malgrange [34] proved in 1983 that if $f(x_1,\dots,x_n)\in\mathbf{C}[x_1,\dots,x_n]$ and s_0 is a root of the Bernstein–Sato polynomial of f, then $e^{2\pi i s_0}$ is a monodromy eigenvalue of f. Therefore Conjecture 0.4 implies Conjecture 0.2.

The above conjectures were verified by Loeser for polynomials in two variables [31] and for non-degenerate polynomials in several variables subject to extra non-natural technical conditions (see [32] or Theorem 0.35). In higher dimension or in a more general setting, there are various partial results, e.g., [3, 4, 6, 7, 22, 25, 29, 30, 32, 35, 41, 43, 45].

In [29] Lemahieu and Van Proeyen proved the Monodromy Conjecture for the local topological zeta function (a kind of limit of Igusa zeta functions) of a non-degenerate surface singularity. Hence they achieve the result of Loeser for the topological zeta function in dimension three without the extra conditions. Very recently Takeuchi [40] obtained some higher-dimensional analogues of [29], assuming certain combinatorial restrictions.

0.2. **Statement of the main theorem.** The (first) goal of this paper is to obtain the result of Lemahieu and Van Proeyen for the original local Igusa zeta function, i.e., to prove Conjecture 0.3 for a polynomial in three variables that is non-degenerate over $\mathbf{C}$ and $\mathbf{F}_p$ with respect to its Newton polyhedron. Before formulating our theorem precisely, let us first define the Newton polyhedron of a polynomial and the notion of non-degeneracy.

Definition 0.5 (Newton polyhedron)**.** Let R be a ring. For $\omega=(\omega_1,\dots,\omega_n)\in\mathbf{Z}^n_{\geqslant 0}$, we denote by x^ω the corresponding monomial $x_1^{\omega_1}\cdots x_n^{\omega_n}$ in $R[x_1,\dots,x_n]$. Let $f(x)=f(x_1,\dots,x_n)=\sum_{\omega\in\mathbf{Z}^n_{\geqslant 0}}a_\omega x^\omega$ be a nonzero polynomial over R satisfying $f(0)=0$. Denote the support of f by $\operatorname{supp}(f)=\{\omega\in\mathbf{Z}^n_{\geqslant 0}\mid a_\omega\neq 0\}$. The Newton polyhedron Γ_f of f is then defined as the convex hull in $\mathbf{R}^n_{\geqslant 0}$ of the set

$$\bigcup_{\omega\in\operatorname{supp}(f)}\omega+\mathbf{R}^n_{\geqslant 0}.$$

The global Newton polyhedron Γ_f^{gl} of f is defined as the convex hull of $\operatorname{supp}(f)$. Clearly we have $\Gamma_f=\Gamma_f^{\mathrm{gl}}+\mathbf{R}^n_{\geqslant 0}$.

Notation 0.6. Let f be as in Definition 0.5. For every face[2] τ of the Newton polyhedron Γ_f of f, we put

$$f_\tau(x)=\sum_{\omega\in\tau}a_\omega x^\omega.$$

Definition 0.7 (Non-degenerate over $\mathbf{C}$)**.** Let $f(x)=f(x_1,\dots,x_n)$ be a nonzero polynomial in $\mathbf{C}[x_1,\dots,x_n]$ satisfying $f(0)=0$. We say that f is non-degenerate over $\mathbf{C}$ with respect to all the faces of its Newton polyhedron Γ_f, if for every[3] face τ of Γ_f, the zero locus $f_\tau^{-1}(0)\subseteq\mathbf{C}^n$ of f_τ has no singularities in $(\mathbf{C}^\times)^n$.

We say that f is non-degenerate over $\mathbf{C}$ with respect to all the compact faces of its Newton polyhedron, if the same condition is satisfied, but only for the compact faces τ of Γ_f.

[2]By a face of Γ_f we mean Γ_f itself or one of its proper faces, which are the intersections of Γ_f with a supporting hyperplane. See, e.g., [38].

[3]Thus also for Γ_f.

Definition 0.8 (Non-degenerate over $\mathbf{Q}_p$)**.** Let $f(x) = f(x_1, \ldots, x_n)$ be a nonzero polynomial in $\mathbf{Q}_p[x_1, \ldots, x_n]$ satisfying $f(0) = 0$. We say that f is non-degenerate over $\mathbf{Q}_p$ with respect to all the faces of its Newton polyhedron Γ_f, if for every[3] face τ of Γ_f, the zero locus $f_\tau^{-1}(0) \subseteq \mathbf{Q}_p^n$ of f_τ has no singularities in $(\mathbf{Q}_p^\times)^n$.

We say that f is non-degenerate over $\mathbf{Q}_p$ with respect to all the compact faces of its Newton polyhedron, if we have the same condition, but only for the compact faces τ of Γ_f.

Notation 0.9. For $f \in \mathbf{Z}_p[x_1, \ldots, x_n]$, we denote by $\overline{f}$ the polynomial over $\mathbf{F}_p$, obtained from f, by reducing each of its coefficients modulo $p\mathbf{Z}_p$.

Definition 0.10 (Non-degenerate over $\mathbf{F}_p$)**.** Let $f(x) = f(x_1, \ldots, x_n)$ be a nonzero polynomial in $\mathbf{Z}_p[x_1, \ldots, x_n]$ satisfying $f(0) = 0$. We say that f is non-degenerate over $\mathbf{F}_p$ with respect to all the faces of its Newton polyhedron Γ_f, if for every[3] face τ of Γ_f, the zero locus of the polynomial $\overline{f_\tau}$ has no singularities in $(\mathbf{F}_p^\times)^n$, or, equivalently, the system of polynomial congruences

$$\begin{cases} f_\tau(x) \equiv 0 \bmod p, \\ \dfrac{\partial f_\tau}{\partial x_i}(x) \equiv 0 \bmod p; \quad i = 1, \ldots, n; \end{cases}$$

has no solutions in $(\mathbf{Z}_p^\times)^n$.

We say that f is non-degenerate over $\mathbf{F}_p$ with respect to all the compact faces of its Newton polyhedron, if the same condition is satisfied, but only for the compact faces τ of Γ_f.

Remarks 0.11.
- (i) Let $f(x_1, \ldots, x_n) \in \mathbf{Z}[x_1, \ldots, x_n]$ be a nonzero polynomial satisfying $f(0) = 0$. Suppose that f is non-degenerate over $\mathbf{C}$ with respect to all the (compact) faces of its Newton polyhedron Γ_f. Then f is non-degenerate over $\mathbf{F}_p$ with respect to all the (compact) faces of Γ_f, for almost all p. This is a consequence of the Weak Nullstellensatz.
- (ii) The condition of non-degeneracy is a generic condition in the following sense. Let $\Gamma \subseteq \mathbf{R}^n_{\geqslant 0}$ be a Newton polyhedron. Then almost all[4] polynomials $f(x) \in \mathbf{C}[x_1, \ldots, x_n]$ with $\Gamma_f = \Gamma$ are non-degenerate over $\mathbf{C}$ with respect to all the faces of Γ. (The same is true if we replace $\mathbf{C}$ by $\mathbf{Q}_p$.)

We can now state our main theorem.

Theorem 0.12 (Monodromy Conjecture for Igusa's p-adic local zeta function of a non-degenerate surface singularity)**.** *Let $f(x, y, z) \in \mathbf{Z}[x, y, z]$ be a nonzero polynomial in three variables satisfying $f(0, 0, 0) = 0$, and let $U \subseteq \mathbf{C}^3$ be a neighborhood of the origin. Suppose that f is non-degenerate over $\mathbf{C}$ with respect to all the compact faces of its Newton polyhedron, and let p be a prime number such that f is also non-degenerate over $\mathbf{F}_p$ with respect to the same faces.*[5] *Suppose that s_0 is a pole of the local Igusa zeta function Z_f^0 associated to f. Then $e^{2\pi i \Re(s_0)}$ is an eigenvalue of the local monodromy of f at some point of $f^{-1}(0) \cap U$.*

[4]By 'almost all' we mean the following. Let B be any bounded subset of $\mathbf{R}^n_{\geqslant 0}$ that contains all vertices of Γ. Put $N = \#\mathbf{Z}^n \cap \Gamma \cap B$, and associate to every $f(x) \in \mathbf{C}[x_1, \ldots, x_n]$ with $\Gamma_f = \Gamma$ and $\operatorname{supp}(f) \subseteq B$ an N-tuple containing its coefficients. Then the set of N-tuples corresponding to a non-degenerate polynomial, is Zariski-dense in $\mathbf{C}^N$.

[5]By Remark 0.11(i) this is the case for almost all prime numbers p.

Next we want to state two results of Denef and Hoornaert on Igusa's zeta function for non-degenerate polynomials. To do so, we need some notions that are closely related to Newton polyhedra. We introduce them in the following subsection (see also [5, 14, 15, 25]).

0.3. **Preliminaries on Newton polyhedra.** We gave the definition of a Newton polyhedron in Definition 0.5. Now we introduce some related notions.

Definition 0.13 ($m(k)$)**.** Let R be a ring, and let $f(x) = f(x_1, \ldots, x_n)$ be a nonzero polynomial over R satisfying $f(0) = 0$. For $k \in \mathbf{R}^n_{\geqslant 0}$, we define

$$m(k) = \inf_{x \in \Gamma_f} k \cdot x,$$

where $k \cdot x$ denotes the scalar product of k and x.

The infimum in the definition above is actually a minimum, where the minimum can as well be taken over the global Newton polyhedron Γ_f^{gl} of f, which is a compact set, or even over the finite set $\operatorname{supp}(f)$.

Definition 0.14 (First meet locus)**.** Let f be as in Definition 0.13 and $k \in \mathbf{R}^n_{\geqslant 0}$. We define the first meet locus of k as the set

$$F(k) = \{x \in \Gamma_f \mid k \cdot x = m(k)\},$$

which is always a face of Γ_f.

Definition 0.15 (Primitive vector)**.** A vector $k \in \mathbf{R}^n$ is called primitive if the components of k are integers whose greatest common divisor is one.

Definition 0.16 (Δ_τ)**.** Let f be as in Definition 0.13. For a face τ of Γ_f, we call

$$\Delta_\tau = \{k \in \mathbf{R}^n_{\geqslant 0} \mid F(k) = \tau\}$$

the cone associated to τ. The Δ_τ are the equivalence classes of the equivalence relation $\sim$ on $\mathbf{R}^n_{\geqslant 0}$, defined by

$$k \sim k' \qquad \text{if and only if} \qquad F(k) = F(k').$$

The 'cones' Δ_τ thus form a partition of $\mathbf{R}^n_{\geqslant 0}$:

$$\{\Delta_\tau \mid \tau \text{ is a face of } \Gamma_f\} = \mathbf{R}^n_{\geqslant 0} / \sim .$$

The Δ_τ are in fact relatively open[6] convex cones[7] with a very specific structure, as stated in the following lemma.

Lemma 0.17 (Structure of the Δ_τ)**.** [14, Lemma 2.6]. *Let f be as in Definition 0.13. Let τ be a proper face of Γ_f and let $\tau_1, \ldots, \tau_r$ be the facets[8] of Γ_f that contain τ. Let $v_1, \ldots, v_r$ be the unique primitive vectors in $\mathbf{Z}^n_{\geqslant 0} \setminus \{0\}$ that are perpendicular to $\tau_1, \ldots, \tau_r$, respectively. Then the cone Δ_τ associated to τ is the convex cone*

$$\Delta_\tau = \{\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_r v_r \mid \lambda_j \in \mathbf{R}_{>0}\},$$

and its dimension[9] equals $n - \dim \tau$.

[6] A subset of $\mathbf{R}^n_{\geqslant 0}$ is called relatively open if it is open in its affine closure.

[7] A subset C of $\mathbf{R}^n$ is called a convex cone if it is a convex set and $\lambda x \in C$ for all $x \in C$ and all $\lambda \in \mathbf{R}_{>0}$.

[8] A facet is a face of codimension one.

[9] The dimension of a convex cone is the dimension of its affine hull.

Definition 0.18 (Rational, simplicial, simple)**.** For $v_1, \dots, v_r \in \mathbf{R}^n \setminus \{0\}$, we call

$$\Delta = \operatorname{cone}(v_1, \dots, v_r) = \{\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_r v_r \mid \lambda_j \in \mathbf{R}_{>0}\}$$

the cone strictly positively spanned by the vectors $v_1, \dots, v_r$. When the $v_1, \dots, v_r$ can be chosen from $\mathbf{Z}^n$, we call it a rational cone. If we can choose $v_1, \dots, v_r$ linearly independent over $\mathbf{R}$, then Δ is called a simplicial cone. If Δ is rational and $v_1, \dots, v_r$ can be chosen from a $\mathbf{Z}$-module basis of $\mathbf{Z}^n$, we call Δ a simple cone.

It follows from Lemma 0.17 that the topological closures $\overline{\Delta_\tau}$[10] of the cones Δ_τ form a fan[11] of rational polyhedral cones[12].

Remark 0.19. The function m from Definition 0.13 is linear on each $\overline{\Delta_\tau}$.

We state without proofs the following two lemmas (see, e.g., [14]).

Lemma 0.20 (Simplicial decomposition)**.** *Let Δ be the cone strictly positively spanned by the vectors $v_1, \dots, v_r \in \mathbf{R}^n_{\geqslant 0} \setminus \{0\}$. Then there exists a finite partition of Δ into cones δ_i, such that each δ_i is strictly positively spanned by a $\mathbf{R}$-linearly independent subset of $\{v_1, \dots, v_r\}$. We call such a decomposition a simplicial decomposition of Δ without introducing new rays.*

Lemma 0.21 (Simple decomposition)**.** *Let Δ be a rational simplicial cone. Then there exists a finite partition of Δ into simple cones. (In general, such a decomposition requires the introduction of new rays.)*

Finally, we need the following notion, which is related to the notion of a simple cone.

Definition 0.22 (Multiplicity)**.** Let $v_1, \dots, v_r$ be $\mathbf{Q}$-linearly independent vectors in $\mathbf{Z}^n$. The multiplicity of $v_1, \dots, v_r$, denoted by $\operatorname{mult}(v_1, \dots, v_r)$, is defined as the index of the lattice $\mathbf{Z}v_1 + \cdots + \mathbf{Z}v_r$ in the group of points with integral coordinates in the subspace spanned by $v_1, \dots, v_r$ of the $\mathbf{Q}$-vector space $\mathbf{Q}^n$.

If Δ is the cone strictly positively spanned by $v_1, \dots, v_r$, then we define the multiplicity of Δ as the multiplicity of $v_1, \dots, v_r$, and we denote it by $\operatorname{mult}\Delta$.

The following is well-known (see, e.g., [2, §5.3, Thm. 3.1]).

Proposition 0.23. *Let $v_1, \dots, v_r$ be $\mathbf{Q}$-linearly independent vectors in $\mathbf{Z}^n$. The multiplicity of $v_1, \dots, v_r$ equals the cardinality of the set*

$$\mathbf{Z}^n \cap \left\{ \sum\nolimits_{j=1}^{r} h_j v_j \;\middle|\; h_j \in [0,1) \text{ for } j = 1, \dots, r \right\}.$$

Remark 0.24. Let Δ be as in Definition 0.22. Note that Δ is simple if and only if $\operatorname{mult}\Delta = \operatorname{mult}(v_1, \dots, v_r) = 1$.

0.4. **Theorems of Denef and Hoornaert.**

Notation 0.25. For $k = (k_1, \dots, k_n) \in \mathbf{R}^n$, we denote $\sigma(k) = k_1 + \cdots + k_n$.

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[10] We have $\overline{\Delta_\tau} = \{\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_r v_r \mid \lambda_j \in \mathbf{R}_{\geqslant 0}\} = \{k \in \mathbf{R}^n_{\geqslant 0} \mid F(k) \supseteq \tau\}$.

[11] A fan $\mathcal{F}$ is a finite set of rational polyhedral cones such that every face of a cone in $\mathcal{F}$ is contained in $\mathcal{F}$ and the intersection of each two cones C and C' in $\mathcal{F}$ is a face of both C and C'.

[12] A rational polyhedral cone is a closed convex cone, generated by a finite subset of $\mathbf{Z}^n$.

Theorem 0.26. [12, 13].[13] *Let $f(x_1, \dots, x_n) \in \mathbf{Q}_p[x_1, \dots, x_n]$ be a nonzero polynomial with $f(0, \dots, 0) = 0$, and Φ a Schwartz–Bruhat function on $\mathbf{Q}_p^n$. Let $\tau_1, \dots, \tau_r$ be all the facets of Γ_f, and let $v_1, \dots, v_r$ be the unique primitive vectors in $\mathbf{Z}^n_{\geqslant 0} \setminus \{0\}$ that are perpendicular to $\tau_1, \dots, \tau_r$, respectively. Suppose that f is non-degenerate over $\mathbf{Q}_p$ with respect to all the compact faces of its Newton polyhedron, and suppose that the support of Φ is contained in a small enough neighborhood of the origin. If s_0 is a pole of $Z_{f,\Phi}$, then*

$$(1)\qquad \begin{aligned} s_0 &= -1 + \frac{2k\pi i}{\log p} \qquad \text{for some } k \in \mathbf{Z}, \text{ or} \\ s_0 &= -\frac{\sigma(v_j)}{m(v_j)} + \frac{2k\pi i}{m(v_j)\log p} \end{aligned}$$

for some $j \in \{1, \dots, r\}$ with $m(v_j) \neq 0$ and some $k \in \mathbf{Z}$.

Essential in the proof of Theorem 0.12 is the following combinatorial formula for Z_f^0 for non-degenerate polynomials due to Denef and Hoornaert.

Theorem 0.27. [14, Thm. 4.2]. *Let $f(x) = f(x_1, \dots, x_n)$ be a nonzero polynomial in $\mathbf{Z}_p[x_1, \dots, x_n]$ satisfying $f(0) = 0$. Suppose that f is non-degenerate over $\mathbf{F}_p$ with respect to all the compact faces of its Newton polyhedron Γ_f. Then the local Igusa p-adic zeta function associated to f is the meromorphic complex function*

$$(2)\qquad Z_f^0 = \sum_{\substack{\tau \text{ compact} \\ \text{face of } \Gamma_f}} L_\tau S(\Delta_\tau),$$

with

$$L_\tau : s \mapsto L_\tau(s) = \left(\frac{p-1}{p}\right)^n - \frac{N_\tau}{p^{n-1}} \frac{p^s - 1}{p^{s+1} - 1},$$
$$N_\tau = \# \left\{ x \in (\mathbf{F}_p^\times)^n \;\middle|\; \overline{f_\tau}(x) = 0 \right\},$$

and

$$S(\Delta_\tau) : s \mapsto S(\Delta_\tau)(s) = \sum_{k \in \mathbf{Z}^n \cap \Delta_\tau} p^{-\sigma(k) - m(k)s}$$

for every compact face τ of Γ_f.

The $S(\Delta_\tau)$ can be calculated as follows. Choose a decomposition $\{\delta_i\}_{i \in I}$ of the cone Δ_τ into simplicial cones δ_i without introducing new rays. Then clearly

$$(3)\qquad S(\Delta_\tau) = \sum_{i \in I} S(\delta_i),$$

in which

$$S(\delta_i) : s \mapsto S(\delta_i)(s) = \sum_{k \in \mathbf{Z}^n \cap \delta_i} p^{-\sigma(k) - m(k)s}.$$

Suppose that the cone δ_i is strictly positively spanned by the linearly independent primitive vectors v_j, $j \in J_i$, in $\mathbf{Z}^n_{\geqslant 0} \setminus \{0\}$. Then we have

$$S(\delta_i)(s) = \frac{\Sigma(\delta_i)(s)}{\prod_{j \in J_i} (p^{\sigma(v_j) + m(v_j)s} - 1)},$$

[13] The theorem was announced in [12] and a proof is written down in [13].

with $\Sigma(\delta_i)$ *the function*

$$\Sigma(\delta_i) : s \mapsto \Sigma(\delta_i)(s) = \sum_h p^{\sigma(h)+m(h)s}, \tag{4}$$

where h *runs through the elements of the set*

$$H(v_j)_{j\in J_i} = \mathbf{Z}^n \cap \lozenge(v_j)_{j\in J_i},$$

with

$$\lozenge(v_j)_{j\in J_i} = \left\{ \sum\nolimits_{j\in J_i} h_j v_j \;\middle|\; h_j \in [0,1) \text{ for all } j \in J_i \right\}$$

the fundamental parallelepiped spanned by the vectors v_j, $j \in J_i$.

Remark 0.28. There exists a global version of this formula for Z_f; the condition is that f is non-degenerate over $\mathbf{F}_p$ with respect to <u>all</u> the faces of its Newton polyhedron, and the sum (2) should be taken over <u>all</u> the faces of Γ_f as well (including Γ_f itself). In the few definitions that follow, we state everything for the local Igusa zeta function Z_f^0, since this zeta function is the subject of our theorem. Nevertheless, all notions and results have straightforward analogues for Z_f (see [14]).

The formula for Z_f^0 in the theorem confirms (under slightly different conditions) the result of Denef that if s_0 is a pole of Z_f^0, it must be one of the Numbers (1) from Theorem 0.26. We call these numbers the candidate poles of Z_f^0.

0.5. **Expected order and contributing faces.**

Definition 0.29 (Expected order of a candidate pole)**.** Let f be as in Theorem 0.27, and suppose that s_0 is a candidate pole of Z_f^0. We define the expected order of the candidate pole s_0 (as a pole of Z_f^0 with respect to the formula in Theorem 0.27) as

$$\max\{\text{order of } s_0 \text{ as a pole of } L_\tau S(\Delta_\tau) \mid \tau \text{ face of } \Gamma_f\}. \tag{5}$$

Hereby we agree that the order of s_0 as a pole of $L_\tau S(\Delta_\tau)$ equals zero, if s_0 is not a pole of $L_\tau S(\Delta_\tau)$. Note that if $\Re(s_0) \neq -1$, we may omit L_τ in (5).

Remark 0.30. Clearly the expected order of a candidate pole s_0 of Z_f^0 is an upper bound for the actual order of s_0 as a pole of Z_f^0.

Definition 0.31 (Contributing vector/face/cone)**.** Let f be as in Theorem 0.27, and suppose that s_0 is a candidate pole of Z_f^0. We say that a primitive vector $v \in \mathbf{Z}^n_{\geqslant 0} \setminus \{0\}$ contributes to s_0 if $p^{\sigma(v)+m(v)s_0} = 1$, or, equivalently, if

$$s_0 = -\frac{\sigma(v)}{m(v)} + \frac{2k\pi i}{m(v)\log p}$$

for some $k \in \mathbf{Z}$. We say that a facet τ of Γ_f contributes to the candidate pole s_0, if the unique primitive vector $v \in \mathbf{Z}^n_{\geqslant 0} \setminus \{0\}$ that is perpendicular to τ, contributes to s_0. More generally, a face of Γ_f is said to contribute to s_0, if it is contained in one or more contributing facets of Γ_f. Finally, we say that a cone $\delta = \text{cone}(v_1, \ldots, v_r)$, minimally[14] strictly positively spanned by the primitive vectors $v_1, \ldots, v_r \in \mathbf{Z}^n_{\geqslant 0} \setminus \{0\}$, contributes to s_0, if one or more of the vectors v_j contribute to s_0. Note that in this way a face τ of Γ_f contributes to s_0 if and only if its associated cone Δ_τ does so.

[14]By 'minimally' we mean that $\delta \neq \text{cone}(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_r)$ for all $j \in \{1, \ldots, r\}$.

Let f be as above, and suppose that s_0 is a candidate pole of Z_f^0 with $\Re(s_0) \neq -1$. From Theorem 0.27 it should be clear that if we want to investigate whether s_0 is actually a pole or not, we only need to consider the sum $\sum L_\tau S(\Delta_\tau)$ over the contributing compact faces τ of Γ_f. Furthermore, if, for a contributing compact face τ, in order to deal with $S(\Delta_\tau)$, we consider a simplicial subdivision $\{\delta_i\}_i$ of the cone Δ_τ, we only need to take into account the terms of (3) corresponding to the contributing simplicial cones in $\{\delta_i\}_i$, in order to decide whether s_0 is a pole or not.

Remark 0.32. Let f be as in Theorem 0.27. Let τ be a facet of Γ_f and s_0 a candidate pole of Z_f^0. One easily checks that if τ contributes to s_0, then $\Re(s_0) = -1/t_0$, where $(t_0, \dots, t_0)$ denotes the intersection point of the affine support $\operatorname{aff}\tau$ of τ with the diagonal $\{(t, \dots, t) \mid t \in \mathbf{R}\} \subseteq \mathbf{R}^n$ of the first orthant (see also [14, Prop. 5.1]).

0.6. **B_1-facets and the structure of the proof of the main theorem.** The proof of Theorem 0.12 consists of three results, namely Theorem 0.34, Proposition 0.39, and Theorem 0.40, all stated below. The first two results have been proved by Lemahieu and Van Proeyen in [29]; the last one is the subject of the current paper; its proof covers Sections 1–8 (pp. 18–97). In order to state the theorems, we have one last important notion to introduce: that of a B_1-facet.

Definition 0.33 (B_1-facet)**.** Let R be a ring and $n \in \mathbf{Z}_{\geqslant 2}$. Let $f(x) = f(x_1, \dots, x_n)$ be a nonzero polynomial over R satisfying $f(0) = 0$. We call a facet τ of Γ_f a B_1-simplex for a variable $x_i \in \{x_1, \dots, x_n\}$, if τ is a simplex with $n-1$ vertices in the coordinate hyperplane $\{x_i = 0\}$ and one vertex in the hyperplane $\{x_i = 1\}$. We call a facet of Γ_f a B_1-simplex, if it is a B_1-simplex for some variable x_i.

A facet τ of Γ_f is called non-compact for a variable $x_j \in \{x_1, \dots, x_n\}$, if for every point $(x_1, \dots, x_n) \in \tau$, we have $(x_1, \dots, x_{j-1}, x_j + 1, x_{j+1}, \dots, x_n) \in \tau$. For $j \in \{1, \dots, n\}$, we shall denote by π_j the projection

$$\pi_j : \mathbf{R}^n \to \mathbf{R}^{n-1} : (x_1, \dots, x_n) \mapsto (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n).$$

Suppose that $n \geqslant 3$. We call a facet τ of Γ_f a non-compact B_1-facet for a variable x_i, if τ is non-compact for precisely one variable $x_j \neq x_i$ and $\pi_j(\tau)$ is a B_1-simplex in $\mathbf{R}^{n-1}$ for the variable x_i. A facet of Γ_f is called a non-compact B_1-facet, if it is a non-compact B_1-facet for some variable x_i.

Finally, we call a facet of Γ_f a B_1-facet (or B_1 for short) for a variable x_i, if it is either a B_1-simplex for x_i or a non-compact B_1-facet for x_i; we call it a B_1-facet when it is B_1 for some variable x_i.

The first step in the proof of Theorem 0.12 is the fact that 'almost all' candidate poles of Z_f^0 induce monodromy eigenvalues; 'almost all' means all, except—possibly—those that are only contributed by B_1-facets.

Theorem 0.34 (On the candidate poles of Z_f^0 contributed by non-B_1-facets)**.** *Cfr.* [29, Theorem 10]. *Let f and p be as in Theorem 0.12. Let s_0 be a candidate pole of Z_f^0 and suppose that s_0 is contributed by a facet of Γ_f that is not a B_1-facet. Then $e^{2\pi i \Re(s_0)}$ is an eigenvalue of the local monodromy of f at some point of the surface $f^{-1}(0) \subseteq \mathbf{C}^3$ close to the origin.*

The proof of the theorem above relies on Varchenko's formula [42] for the zeta function of monodromy of f (at the origin) in terms of the Newton polyhedron of f, which in turn relies on A'Campo's formula [1] for the same zeta function in terms of an embedded resolution of singularities of $f^{-1}(0) \subseteq \mathbf{C}^3$.

In this context, we would also like to mention the results of Denef–Sperber [21] and Cluckers [8, 9] on exponential sums associated to non-degenerate polynomials. Here one also obtains nice results when imposing certain conditions on the faces of the Newton polyhedron that are similar to the one in the theorem above.

This is probably also a good place to state the result of Loeser on the Monodromy Conjecture for non-degenerate singularities. Loeser proves (in general dimension) a result similar to Theorem 0.34, imposing several, rather technical conditions on the Newton polyhedron's facets.

Theorem 0.35. [32]. *Let $f(x_1,\ldots,x_n) \in \mathbf{C}[x_1,\ldots,x_n]$ be a nonzero polynomial with $f(0,\ldots,0) = 0$. Suppose that f is non-degenerate over $\mathbf{C}$ with respect to all the compact faces of its Newton polyhedron. Let τ_0 be a compact facet of Γ_f, and let $\tau_1,\ldots,\tau_r$ be all the facets of Γ_f that are different, but not disjoint from τ_0. Denote by $v_0, v_1, \ldots, v_r$ the unique primitive vectors in $\mathbf{Z}^n_{\geqslant 0} \setminus \{0\}$ that are perpendicular to $\tau_0, \tau_1, \ldots, \tau_r$, respectively. Suppose that*

(i) $\dfrac{\sigma(v_0)}{m(v_0)} < 1$ *and that*

(ii) $\dfrac{1}{\mathrm{mult}(v_0,v_j)}\left(\sigma(v_j) - \dfrac{\sigma(v_0)}{m(v_0)}m(v_j)\right) \notin \mathbf{Z}$ *for all $j \in \{1,\ldots,r\}$.*

Then $-\sigma(v_0)/m(v_0)$ is a root of the local Bernstein–Sato polynomial b_f^0 of f. Hereby $\mathrm{mult}(v_0,v_j)$ *denotes the multiplicity of v_0 and v_j (cfr. Definition 0.22).*

By the result of Malgrange [34] we mentioned earlier, under the conditions of the theorem, we also have that $e^{-2\pi i\sigma(v_0)/m(v_0)}$ is an eigenvalue of the local monodromy of f at some point of $f^{-1}(0) \subseteq \mathbf{C}^n$ close to the origin. Loeser proves that this remains true if we replace Condition (i) by $\sigma(v_0)/m(v_0) \notin \mathbf{Z}$.

Let us go back to Theorem 0.34 and the B_1-facets. What can we say about the candidate poles of Z_f^0 that are exclusively contributed by B_1-facets? In 1984 Denef announced the following theorem (in general dimension) on candidate poles of $Z_{f,\Phi}$ that are contributed by a single B_1-simplex.

Theorem 0.36. *Cfr.* [12, 20].[15] *Let $f(x_1,\ldots,x_n) \in \mathbf{Q}_p[x_1,\ldots,x_n]$ be a nonzero polynomial with $f(0,\ldots,0) = 0$, and Φ a Schwartz–Bruhat function on $\mathbf{Q}_p^n$. Suppose that f is non-degenerate over $\mathbf{Q}_p$ with respect to all the compact faces of its Newton polyhedron, and suppose that the support of Φ is contained in a small enough neighborhood of the origin. Let $\tau_0, \tau_1, \ldots, \tau_r$ be all the facets of Γ_f, and let $v_0, v_1, \ldots, v_r$ be the unique primitive vectors in $\mathbf{Z}^n_{\geqslant 0} \setminus \{0\}$ that are perpendicular to $\tau_0, \tau_1, \ldots, \tau_r$, respectively. Suppose that τ_0 is a B_1-simplex, that $\sigma(v_0)/m(v_0) \neq 1$, and that $\sigma(v_0)/m(v_0) \neq \sigma(v_j)/m(v_j)$ for all $j \in \{1,\ldots,r\}$. Then there is no pole s_0 of $Z_{f,\Phi}$ with $\Re(s_0) = -\sigma(v_0)/m(v_0)$.*

We can restate Denef's theorem as follows.

Theorem 0.37. *Cfr.* [12, 20]. *Let $f(x_1,\ldots,x_n) \in \mathbf{Q}_p[x_1,\ldots,x_n]$ be a nonzero polynomial with $f(0,\ldots,0) = 0$, and Φ a Schwartz–Bruhat function on $\mathbf{Q}_p^n$. Suppose that f is non-degenerate over $\mathbf{Q}_p$ with respect to all the compact faces of its*

[15]The theorem was announced in [12] and a proof is sketched in the real case in [20]. This proof is adaptable to the p-adic case, but except for dimension three, a complete detailed proof has not been written down yet.

Newton polyhedron, and suppose that the support of Φ is contained in a small enough neighborhood of the origin. Let $s_0 \neq -1$ be a real candidate pole of $Z_{f,\Phi}$. Suppose that exactly one facet of Γ_f contributes to s_0 and that this facet is a B_1-simplex. Then there exists no pole of $Z_{f,\Phi}$ with real part s_0.

Denef noticed that one cannot expect this theorem to be generally true for candidate poles that are contributed by several B_1-simplices. He gave the following counterexample[16] in dimension three. We will discuss the example in detail, as it also illustrates Denef and Hoornaert's formula.

Example 0.38 (Actual pole of Z_f^0 only contributed by B_1-facets)**.** [Denef, 1984]. Let $p \geqslant 3$ be a prime number and consider $f = x^3 + xy + y^2 + z^2 \in \mathbf{Z}_p[x, y, z]$. The Newton polyhedron Γ_f of f and the cones associated to its faces are drawn in Figure 1. One checks that f is non-degenerate over $\mathbf{F}_p$ with respect to all the compact faces of Γ_f ($p \neq 2$).

Table 1 gives an overview of the facets τ_j of Γ_f, their associated numerical data $(m(v_j), \sigma(v_j))$, and their associated candidate poles of Z_f^0. Facets τ_0 and τ_1 are B_1-simplices, while τ_2, τ_3, τ_4 lie in coordinate hyperplanes and hence do not yield any candidate poles. Poles of Z_f^0 are therefore among the numbers

$$s_k = -\frac{3}{2} + \frac{k\pi i}{3\log p}, \quad s'_l = -1 + \frac{2l\pi i}{\log p}; \qquad k, l \in \mathbf{Z}.$$

The candidate poles s_k with $3 \nmid k$ are only contributed by τ_0 and have expected order one, while the s_k with $3 \mid k$ are contributed by τ_0 and τ_1; the latter have expected order two since the contributing facets τ_0 and τ_1 share the edge $[BD]$.

We will now calculate Z_f^0 using Theorem 0.27 in order to find out which candidate poles are actually poles. Table 2 provides an overview of Γ_f's compact faces and their associated cones and all the data needed to fill in the theorem's formula. The numbers N_τ that appear in the $L_\tau(s)$ are listed in the third column. Hereby N_0 and N_1 represent the numbers

$$N_0 = \#\left\{(x,z) \in (\mathbf{F}_p^\times)^2 \;\middle|\; x^3 + z^2 = 0\right\} \qquad \text{and}$$
$$N_1 = \#\left\{(y,z) \in (\mathbf{F}_p^\times)^2 \;\middle|\; y^2 + z^2 = 0\right\}.$$

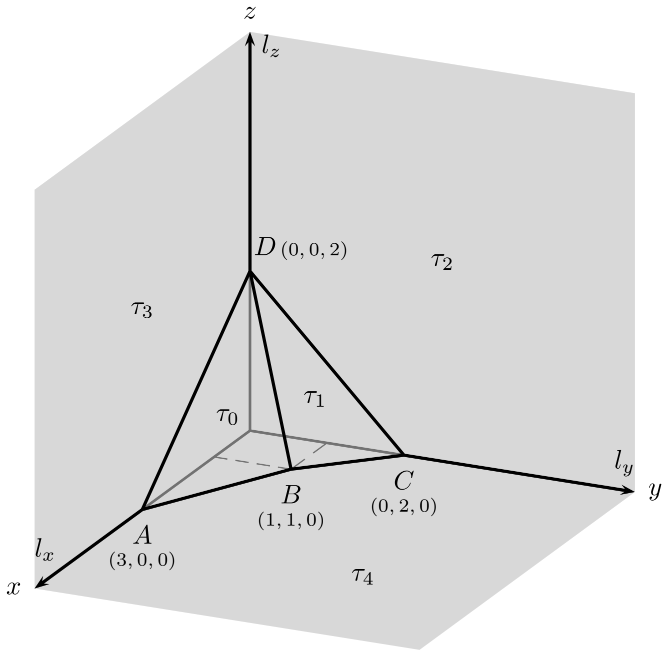
The $S(\Delta_\tau)(s)$ can be calculated based on the data on the cones Δ_τ in the right-hand side of Table 2. We find that Δ_τ is simplicial for every (compact) face τ of Γ_f, except for $\tau = D$. Those cones Δ_τ, $\tau \neq D$, are even simple, except for $\Delta_A, \Delta_B, \Delta_{[AB]}$, whose corresponding fundamental parallelepipeds contain besides the origin also the integral point $(1, 2, 2)$ (see Table 1). In order to calculate $S(\Delta_D)(s)$, we choose to decompose Δ_D into the simplicial cones $\delta_1, \delta_2, \delta_3$ that happen to be simple as well (see Table 2).

We now obtain Z_f^0 as

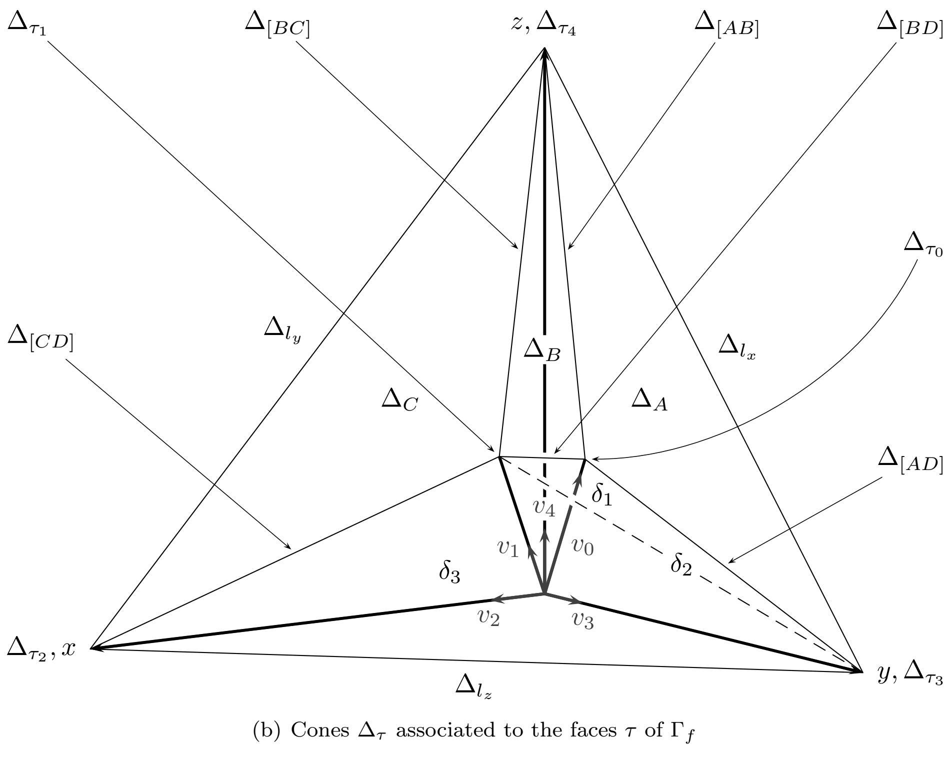
$$Z_f^0(s) = \sum_{\substack{\tau \text{ compact} \\ \text{face of } \Gamma_f}} L_\tau(s) S(\Delta_\tau)(s) = \frac{(p-1)(p^{s+3}-1)}{p^3(p^{s+1}-1)(p^{2s+3}-1)}.$$

Note that $Z_f^0(s)$ does not depend on N_0 or N_1. We conclude that the candidate poles that are only contributed by a single B_1-simplex are not poles. On the other

[16]Denef in fact showed that for $f = x^n + xy + y^m + z^2$, the candidate pole $-3/2$ (which is contributed by two B_1-simplices) is an actual pole of $Z_{f,\Phi}$ for n, m big enough.



(a) Newton polyhedron Γ_f of $f = x^3 + xy + y^2 + z^2$ and its faces



(b) Cones Δ_τ associated to the faces τ of Γ_f

FIGURE 1. Combinatorial data associated to $f = x^3 + xy + y^2 + z^2$

TABLE 1. Numerical data associated to $(1,2,2)$ and the facets of Γ_f

facet τ of Γ_f	τ compact?	primitive vector $v \perp \tau$	$m(v)$	$\sigma(v)$	candidate poles of Z_f^0 contributed by τ
τ_0	yes	$v_0(2,4,3)$	6	9	$-\dfrac{3}{2}+\dfrac{k\pi i}{3\log p};\ k \in \mathbf{Z}$
τ_1	yes	$v_1(1,1,1)$	2	3	$-\dfrac{3}{2}+\dfrac{k\pi i}{\log p};\ k \in \mathbf{Z}$
τ_2	no	$v_2(1,0,0)$	0	1	none
τ_3	no	$v_3(0,1,0)$	0	1	none
τ_4	no	$v_4(0,0,1)$	0	1	none
		integral point h	$m(h)$	$\sigma(h)$	
		$(1,2,2)$	3	5	

hand we find that the numbers s_{3k}, despite being only contributed by B_1-simplices, are indeed poles, although their order is lower than expected.

In situations as in Example 0.38 that are not covered by Theorem 0.34, one needs to prove that the pole in question induces a monodromy eigenvalue. This is done by Lemahieu and Van Proeyen in the following proposition and forms the second step in the proof of Theorem 0.12. Note that the two B_1-simplices in the example are B_1 with respect to different variables.

Proposition 0.39. *Cfr.* [29, Theorem 15]. *Let f and p be as in Theorem 0.12. Let s_0 be a candidate pole of Z_f^0 and suppose that s_0 is contributed by two B_1-facets of Γ_f that are <u>not</u> B_1 for a same variable and that have an edge in common. Then $e^{2\pi i \Re(s_0)}$ is an eigenvalue of the local monodromy of f at some point of the surface $f^{-1}(0) \subseteq \mathbf{C}^3$ close to the origin.*

The proof of the proposition again uses Varchenko's formula and is part of the proof of Theorem 15 in [29]. In this paper one considers the local topological zeta function $Z_f^{\mathrm{top},0}$ instead of Z_f^0; however, the candidate poles of $Z_f^{\mathrm{top},0}$ are precisely the real parts of the candidate poles of Z_f^0, and whenever a facet of Γ_f contributes to a candidate pole s_0 of Z_f^0, it contributes to the candidate pole $\Re(s_0)$ of $Z_f^{\mathrm{top},0}$ as well; therefore Proposition 0.39 follows in the same way.

In order to conclude Theorem 0.12, we want to prove that the remaining candidate poles, i.e., candidate poles only contributed by B_1-facets, but not satisfying the conditions of Proposition 0.39, are actually not poles. The result is—under slightly different conditions and in dimension three—an optimization of Theorem 0.37, partially allowing the candidate pole s_0 to be contributed by several B_1-facets, including non-compact ones. This is the final step of the proof.

Theorem 0.40 (On candidate poles of Z_f^0 only contributed by B_1-facets)**.** *Let $f(x,y,z) \in \mathbf{Z}_p[x,y,z]$ be a nonzero polynomial in three variables with $f(0,0,0)=0$. Suppose that f is non-degenerate over $\mathbf{F}_p$ with respect to all the compact faces of its Newton polyhedron. Let s_0 be a candidate pole of Z_f^0 with $\Re(s_0) \neq -1$, and suppose*

Table 2. Data associated to the compact faces of Γ_f and their associated cones

face τ of Γ_f	$\overline{f_\tau}$	N_τ	$L_\tau(s)$	cone Δ_τ, δ_i	dim Δ_τ, δ_i	primitive generators	mult Δ_τ, δ_i	$S(\Delta_\tau)(s), S(\delta_i)(s)$
A	x^3	0	$\left(\frac{p-1}{p}\right)^3$	Δ_A	3	v_0, v_3, v_4	2	$\frac{1+p^{3s+5}}{(p^{6s+9}-1)(p-1)^2}$
B	xy	0	$\left(\frac{p-1}{p}\right)^3$	Δ_B	3	v_0, v_1, v_4	2	$\frac{1+p^{3s+5}}{(p^{6s+9}-1)(p^{2s+3}-1)(p-1)}$
C	y^2	0	$\left(\frac{p-1}{p}\right)^3$	Δ_C	3	v_1, v_2, v_4	1	$\frac{1}{(p^{2s+3}-1)(p-1)^2}$
				δ_1	3	v_0, v_1, v_3	1	$\frac{1}{(p^{6s+9}-1)(p^{2s+3}-1)(p-1)}$
				δ_2	2	v_1, v_3	1	$\frac{1}{(p^{2s+3}-1)(p-1)}$
				δ_3	3	v_1, v_2, v_3	1	$\frac{1}{(p^{2s+3}-1)(p-1)^2}$
D	z^2	0	$\left(\frac{p-1}{p}\right)^3$	Δ_D	3	v_0, v_1, v_2, v_3	–	$\frac{p^{6s+10}-1}{(p^{6s+9}-1)(p^{2s+3}-1)(p-1)^2}$
$[AB]$	x^3+xy	$(p-1)^2$	$\left(\frac{p-1}{p}\right)^3-\left(\frac{p-1}{p}\right)^2\frac{p^s-1}{p^{s+1}-1}$	$\Delta_{[AB]}$	2	v_0, v_4	2	$\frac{1+p^{3s+5}}{(p^{6s+9}-1)(p-1)}$
$[BC]$	$xy+y^2$	$(p-1)^2$	$\left(\frac{p-1}{p}\right)^3-\left(\frac{p-1}{p}\right)^2\frac{p^s-1}{p^{s+1}-1}$	$\Delta_{[BC]}$	2	v_1, v_4	1	$\frac{1}{(p^{2s+3}-1)(p-1)}$
$[AD]$	x^3+z^2	$(p-1)N_0$	$\left(\frac{p-1}{p}\right)^3-\frac{(p-1)N_0}{p^2}\frac{p^s-1}{p^{s+1}-1}$	$\Delta_{[AD]}$	2	v_0, v_3	1	$\frac{1}{(p^{6s+9}-1)(p-1)}$
$[BD]$	$xy+z^2$	$(p-1)^2$	$\left(\frac{p-1}{p}\right)^3-\left(\frac{p-1}{p}\right)^2\frac{p^s-1}{p^{s+1}-1}$	$\Delta_{[BD]}$	2	v_0, v_1	1	$\frac{1}{(p^{6s+9}-1)(p^{2s+3}-1)}$
$[CD]$	y^2+z^2	$(p-1)N_1$	$\left(\frac{p-1}{p}\right)^3-\frac{(p-1)N_1}{p^2}\frac{p^s-1}{p^{s+1}-1}$	$\Delta_{[CD]}$	2	v_1, v_2	1	$\frac{1}{(p^{2s+3}-1)(p-1)}$
τ_0	x^3+xy+z^2	$(p-1)^2-N_0$	$\left(\frac{p-1}{p}\right)^3-\frac{(p-1)^2-N_0}{p^2}\frac{p^s-1}{p^{s+1}-1}$	Δ_{τ_0}	1	v_0	1	$\frac{1}{p^{6s+9}-1}$
τ_1	$xy+y^2+z^2$	$(p-1)^2-N_1$	$\left(\frac{p-1}{p}\right)^3-\frac{(p-1)^2-N_1}{p^2}\frac{p^s-1}{p^{s+1}-1}$	Δ_{τ_1}	1	v_1	1	$\frac{1}{p^{2s+3}-1}$

that s_0 is only contributed by B_1-facets of Γ_f. Assume also that for any pair of contributing B_1-facets, we have that

- *either they are B_1-facets for a same variable,*
- *or they have at most one point in common.*

Then s_0 is not a pole of Z_f^0.

This is the key theorem of the paper that we will prove in the next eight sections. The theorem has been proved for the local topological zeta function by Lemahieu and Van Proeyen [29, Proposition 14]. In our proof we will consider the same seven cases as they did, distinguishing all possible configurations of contributing B_1-facets. The idea is to calculate in every case the residue(s) of Z_f^0 in the candidate pole in question s_0, based on Denef and Hoornaert's formula for Z_f^0 for non-degenerate f (Theorem 0.27).

The main difficulty in comparison with the topological zeta function approach lies in the calculation of $\Sigma(\delta)(s_0)$ and $\Sigma(\delta)'(s_0)$ for different simplicial cones δ (see Equation (4) in Theorem 0.27). Where for the topological zeta function it is sufficient to consider the multiplicity of a cone, for the p-adic zeta function one has to sum over the lattice points that yield this multiplicity. Lemahieu and Van Proeyen used a computer algebra package to manipulate the rational expressions they obtained for the topological zeta function and so achieved their result; in the p-adic case this approach is no longer possible.

In order to deal with the aforementioned sums, we study in Section 1 the integral points in a three-dimensional fundamental parallelepiped. The aim is to achieve an explicit description of the points that we can use in the rest of the proof to calculate the sums $\Sigma(\delta)(s_0)$ and $\Sigma(\delta)'(s_0)$ over those points. These calculations often lead to polynomial expressions with floor or ceil functions in the exponents; dealing with them forms the second main difficulty of the proof. A third complication is due to the existence of imaginary candidate poles; their residues are usually harder to calculate than those of their real colleagues.

Remark 0.41. Although everything in this paper is formulated for $\mathbf{Q}_p$, the results can be generalized in a straightforward way to arbitrary p-adic fields. The reason is that Denef and Hoornaert's formula for Igusa's zeta function has a very similar form over finite field extensions of $\mathbf{Q}_p$.

0.7. **Overview of the paper.** As mentioned before, Section 1 contains an elaborate study of the integral points in three-dimensional fundamental parallelepipeds.

Sections 2–8 cover the proof of Theorem 0.40; every section treats one possible configuration of B_1-facets contributing to a same candidate pole.

In Section 9 we verify the analogue of Theorem 0.40 for Igusa's zeta function of a polynomial $f(x_1,\ldots,x_n) \in \mathbf{Z}_p[x_1,\ldots,x_n]$ and a non-trivial character of $\mathbf{Z}_p^\times$. This leads to the Monodromy Conjecture in this case as well.

In Section 10 we state and prove the motivic version of our main theorem; i.e., we obtain the motivic Monodromy Conjecture for a non-degenerate surface singularity. This section also contains a detailed proof of the motivic analogue of Denef and Hoornaert's formula. Our objective is to obtain a formula for the motivic zeta function as an element in the ring $\mathcal{M}_{\mathbf{C}}[[T]]$, as it is defined, rather than as an element in some localization of $\mathcal{M}_{\mathbf{C}}[[T]]$.[17] This explains the technicality of the formula and its proof.

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1. On the integral points in a three-dimensional fundamental parallelepiped spanned by primitive vectors

1.0. Introduction.

We recall some basic definitions and results.

Definition 1.1 (Primitive vector)**.** A vector $w = (a_1, \dots, a_n) \in \mathbf{Z}^n$ is called primitive if $\gcd(a_1, \dots, a_n) = 1$.

Definition 1.2 (Fundamental parallelepiped)**.** Let $w_1, \dots, w_t$ be $\mathbf{R}$-linear independent primitive vectors in $\mathbf{Z}^n_{\geqslant 0}$. We call the set

$$\Diamond(w_1, \dots, w_t) = \left\{ \sum\nolimits_{j=1}^{t} h_j w_j \;\middle|\; h_j \in [0,1); \quad j = 1, \dots, t \right\} \subseteq \mathbf{R}^n_{\geqslant 0}$$

the fundamental parallelepiped spanned by the vectors $w_1, \dots, w_t$.

Definition 1.3 (Multiplicity)**.** The number of integral points (i.e., points with integer coordinates) in a fundamental parallelepiped $\Diamond(w_1, \dots, w_t)$ is called the multiplicity of the fundamental parallelepiped. We denote it by

$$\operatorname{mult} \Diamond(w_1, \dots, w_t) = \# \left(\mathbf{Z}^n \cap \Diamond(w_1, \dots, w_t) \right).$$

The following result is well-known (see, e.g., [2, §5.3, Thm. 3.1]).

Proposition 1.4. *Let $w_1, \dots, w_t$ be $\mathbf{R}$-linear independent primitive vectors in $\mathbf{Z}^n_{\geqslant 0}$. The multiplicity of the fundamental parallelepiped $\Diamond(w_1, \dots, w_t)$ equals the greatest common divisor of the absolute values of the determinants of all $(t \times t)$-submatrices of the $(t \times n)$-matrix whose rows contain the coordinates of $w_1, \dots, w_t$.*

Notation 1.5. We will denote the determinant of a square, real matrix $M = (a_{ij})_{ij}$ by $\det M = |a_{ij}|_{ij}$ and its absolute value by $|\det M| = \|a_{ij}\|_{ij}$.

For the rest of this section, we fix three linearly independent primitive vectors $w_1, w_2, w_3 \in \mathbf{Z}^3_{\geqslant 0}$ and denote their coordinates by

$$w_1(a_1, b_1, c_1), \qquad w_2(a_2, b_2, c_2), \qquad \text{and} \qquad w_3(a_3, b_3, c_3).$$

We also fix notations for the following sets and their cardinalities (cfr. Figure 2):

$$\begin{aligned}
H &= \mathbf{Z}^3 \cap \Diamond(w_1, w_2, w_3), & \mu &= \#H = \operatorname{mult} \Diamond(w_1, w_2, w_3),\\
H_1 &= \mathbf{Z}^3 \cap \Diamond(w_2, w_3) \subseteq H, & \mu_1 &= \#H_1 = \operatorname{mult} \Diamond(w_2, w_3),\\
H_2 &= \mathbf{Z}^3 \cap \Diamond(w_1, w_3) \subseteq H, & \mu_2 &= \#H_2 = \operatorname{mult} \Diamond(w_1, w_3),\\
H_3 &= \mathbf{Z}^3 \cap \Diamond(w_1, w_2) \subseteq H, & \mu_3 &= \#H_3 = \operatorname{mult} \Diamond(w_1, w_2).
\end{aligned}$$

[17] The ring $\mathcal{M}_{\mathbf{C}}$ denotes the localization of the Grothendieck ring of complex algebraic varieties with respect to the class of the affine line, while T is a formal indeterminate.

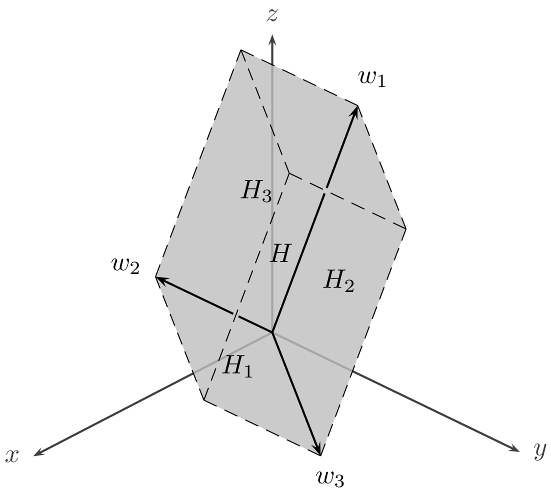


FIGURE 2. A fundamental parallelepiped spanned by three primitive vectors in $\mathbf{Z}^3_{\geqslant 0}$. The sets H, H_1, H_2, H_3 denote the intersections of the respective fundamental parallelepipeds with $\mathbf{Z}^3$.

Throughout this section we consider the matrix

$$M = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \in \mathbf{Z}^{3\times 3}_{\geqslant 0}$$

and its minors

$$\begin{aligned} M_{11} &= \begin{pmatrix} b_2 & c_2 \\ b_3 & c_3 \end{pmatrix}, & M_{12} &= \begin{pmatrix} a_2 & c_2 \\ a_3 & c_3 \end{pmatrix}, & M_{13} &= \begin{pmatrix} a_2 & b_2 \\ a_3 & b_3 \end{pmatrix}, \\ M_{21} &= \begin{pmatrix} b_1 & c_1 \\ b_3 & c_3 \end{pmatrix}, & M_{22} &= \begin{pmatrix} a_1 & c_1 \\ a_3 & c_3 \end{pmatrix}, & M_{23} &= \begin{pmatrix} a_1 & b_1 \\ a_3 & b_3 \end{pmatrix}, \\ M_{31} &= \begin{pmatrix} b_1 & c_1 \\ b_2 & c_2 \end{pmatrix}, & M_{32} &= \begin{pmatrix} a_1 & c_1 \\ a_2 & c_2 \end{pmatrix}, & M_{33} &= \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}. \end{aligned}$$

Let us denote $d = \det M$ and $d_{ij} = \det M_{ij}$; $i, j = 1, 2, 3$. The matrix

$$\operatorname{adj} M = \begin{pmatrix} d_{11} & -d_{21} & d_{31} \\ -d_{12} & d_{22} & -d_{32} \\ d_{13} & -d_{23} & d_{33} \end{pmatrix} = \left((-1)^{i+j} d_{ij}\right)^{\mathrm{T}}_{ij}$$

is called the adjugate matrix of M and has the important property that

$$(\operatorname{adj} M)M = M(\operatorname{adj} M) = dI,$$

with I the (3×3)-identity matrix. According to Proposition 1.4, we have

$$\begin{aligned} \mu &= \#H \ = |d|, \\ \mu_1 &= \#H_1 = \gcd(|d_{11}|, |d_{12}|, |d_{13}|), \\ \mu_2 &= \#H_2 = \gcd(|d_{21}|, |d_{22}|, |d_{23}|), \\ \mu_3 &= \#H_3 = \gcd(|d_{31}|, |d_{32}|, |d_{33}|). \end{aligned}$$

Note that every $h \in H$ can be written in a unique way as

$$h = h_1 w_1 + h_2 w_2 + h_3 w_3$$

with $h_j \in [0, 1)$; $j = 1, 2, 3$. We shall always denote the coordinates of a point $h \in H$ with respect to the basis (w_1, w_2, w_3) of $\mathbf{R}^3$ over $\mathbf{R}$, by (h_1, h_2, h_3).

Notation 1.6. Let $a \in \mathbf{R}$. We denote by $\lfloor a \rfloor$ the largest integer not greater than a (integer part of a) and by $\lceil a \rceil$ the smallest integer not less than a. The fractional part of a will be denoted by $\{a\} = a - \lfloor a \rfloor \in [0, 1)$. By generalization, we shall denote for any $b \in \mathbf{R}_{>0}$ by $\{a\}_b$ the unique element $\{a\}_b \in [0, b)$ such that $a - \{a\}_b \in b\mathbf{Z}$. Note that

$$\{a\}_b = b \left\{ \frac{a}{b} \right\}.$$

The aim of this section is to prove the following theorem.

Theorem 1.7. (i) *The multiplicities μ_1, μ_2, μ_3 all divide μ;*

(ii) *we have even more: for all distinct $i, j \in \{1, 2, 3\}$ it holds that $\mu_i \mu_j \mid \mu$.*

(iii) *For every $h \in H_1$ the coordinates h_2, h_3 of h belong to the set*

$$\left\{ 0, \frac{1}{\mu_1}, \frac{2}{\mu_1}, \ldots, \frac{\mu_1 - 1}{\mu_1} \right\},$$

and every element of the above set is the w_2-coordinate (w_3-coordinate) of exactly one point $h \in H_1$; i.e.,

$$\{h_2 \mid h \in H_1\} = \{h_3 \mid h \in H_1\} = \left\{ 0, \frac{1}{\mu_1}, \frac{2}{\mu_1}, \ldots, \frac{\mu_1 - 1}{\mu_1} \right\}.$$

Moreover, there exists a unique $\xi_1 \in \{0, \ldots, \mu_1 - 1\}$ with $\xi_1 + \mu_1 \mathbf{Z}$ a generator of the additive group $\mathbf{Z}/\mu_1\mathbf{Z}$ (i.e., with $\gcd(\xi_1, \mu_1) = 1$), such that all μ_1 points of H_1 are given by

$$\frac{i}{\mu_1} w_2 + \left\{ \frac{i\xi_1}{\mu_1} \right\} w_3; \qquad i = 0, \ldots, \mu_1 - 1.$$

Of course, we have analogous results for H_2 and H_3.

(iv) *For every $h \in H$ the coordinate h_1 of h belongs to the set*

$$\left\{ 0, \frac{\mu_1}{\mu}, \frac{2\mu_1}{\mu}, \ldots, \frac{\mu - \mu_1}{\mu} \right\}.$$

Moreover, every possible coordinate $l\mu_1/\mu$, $l \in \{0, \ldots, \mu/\mu_1 - 1\}$, occurs precisely μ_1 times. (The set H indeed contains $\mu_1(\mu/\mu_1) = \mu$ points.)

We have of course analogous results for the coordinates h_2 and h_3 of the points $h \in H$.

(v) *By (ii) we can write $\mu = \mu_1 \mu_2 \varphi_3$ with $\varphi_3 \in \mathbf{Z}_{>0}$. It then holds that*

$$\gcd(\mu_1, \mu_2) \mid \mu_3 \mid \gcd(\mu_1, \mu_2)\varphi_3.$$

As a consequence we have that $\gcd(\mu_1, \mu_2, \mu_3) = \gcd(\mu_1, \mu_2)$. (The same result holds, of course, as well for the other two combinations of two out of three multiplicities μ_j.)

(vi) *We give an explicit description of the μ points of H.*

(vii) *Finally, we explain how the numbers ξ_1, ξ_2, ξ_3 (mentioned above), and η, η', l_0 (defined later on) that appear in the several descriptions of points of H, can be calculated from the coordinates of w_1, w_2, w_3.*

1.1. **A group structure on H.**

Notation 1.8. For any $h \in \mathbf{R}^3$ we denote by $\{h\}$ its reduction modulo $\mathbf{Z}w_1 + \mathbf{Z}w_2 + \mathbf{Z}w_3$; i.e., $\{h\}$ denotes the unique element $\{h\} \in \Diamond(w_1, w_2, w_3)$ such that $h - \{h\} \in \mathbf{Z}w_1 + \mathbf{Z}w_2 + \mathbf{Z}w_3$. If we write h as $h = h_1w_1 + h_2w_2 + h_3w_3$ with $h_1, h_2, h_3 \in \mathbf{R}$, we have that

$$\{h\} = \{h_1\}w_1 + \{h_2\}w_2 + \{h_3\}w_3.$$

We can make H into a group by considering addition modulo $\mathbf{Z}w_1 + \mathbf{Z}w_2 + \mathbf{Z}w_3$ as a group law:

$$\{\cdot + \cdot\} : H \times H \to H : (h, h') \mapsto \{h + h'\}.$$

The operation $\{\cdot + \cdot\}$ makes H into a finite abelian group of order μ. It is easy to verify that the subsets H_1, H_2, H_3 of H are in fact subgroups.

Consider the abelian group $\mathbf{Z}^3, +$ and its subgroups

$$\begin{aligned} \Lambda &= \mathbf{Z}w_1 + \mathbf{Z}w_2 + \mathbf{Z}w_3, &\qquad \Lambda_1 &= \mathbf{Z}w_2 + \mathbf{Z}w_3,\\ \Lambda_2 &= \mathbf{Z}w_1 + \mathbf{Z}w_3, & \Lambda_3 &= \mathbf{Z}w_1 + \mathbf{Z}w_2,\end{aligned}$$

generated by $\{w_1, w_2, w_3\}, \{w_2, w_3\}, \{w_1, w_3\}$, and $\{w_1, w_2\}$, respectively. It then holds that

$$\begin{aligned} H &\cong \frac{\mathbf{Z}^3}{\Lambda}, &\qquad H_1 &\cong \frac{\mathbf{Z}^3 \cap (\mathbf{R}w_2 + \mathbf{R}w_3)}{\Lambda_1},\\ H_2 &\cong \frac{\mathbf{Z}^3 \cap (\mathbf{R}w_1 + \mathbf{R}w_3)}{\Lambda_2}, & H_3 &\cong \frac{\mathbf{Z}^3 \cap (\mathbf{R}w_1 + \mathbf{R}w_2)}{\Lambda_3}.\end{aligned} \tag{6}$$

1.2. **Divisibility among the multiplicities μ, μ_1, μ_2, μ_3.**

Since H_1, H_2, H_3 form subgroups of H, their orders divide the order of H: $\mu_1, \mu_2, \mu_3 \mid \mu$ (Theorem 1.7(i)).

Consider the subgroups H_1, H_2 of H. The subgroup $H_1 \cap H_2$ precisely contains the integral points in the fundamental parallelepiped

$$\Diamond(w_3) = \{h_3w_3 \mid h_3 \in [0, 1)\}.$$

Hence since w_3 is primitive, $H_1 \cap H_2$ is the trivial group (this can also be seen from the isomorphisms (6)). It follows that $H_1 + H_2 \cong H_1 \oplus H_2$ and thus

$$|H_1 + H_2| = |H_1 \oplus H_2| = |H_1||H_2| = \mu_1\mu_2.$$

The fact that $H_1 + H_2$ is a subgroup of H now easily implies that $\mu_1\mu_2 \mid \mu$. Analogously, we find that $\mu_1\mu_3, \mu_2\mu_3 \mid \mu$. This proves Theorem 1.7(ii).

From now on, we shall write

$$\mu = \mu_1\mu_2\varphi_3 = \mu_1\mu_3\varphi_2 = \mu_2\mu_3\varphi_1$$

with $\varphi_1, \varphi_2, \varphi_3 \in \mathbf{Z}_{>0}$.

1.3. **On the μ_1 points of H_1.**

Let $h \in H_1 = \mathbf{Z}^3 \cap \Diamond(w_2, w_3)$ and write

$$h = h_2w_2 + h_3w_3$$

with $h_2, h_3 \in [0, 1)$. Because $|H_1| = \mu_1$, the μ_1-th multiple of h in H must equal the identity element:

$$\{\mu_1h\} = \{\mu_1h_2\}w_2 + \{\mu_1h_3\}w_3 = (0, 0, 0);$$

i.e., $\{\mu_1h_2\} = \{\mu_1h_3\} = 0$, and thus $h_2, h_3 \in (1/\mu_1)\mathbf{Z}$.

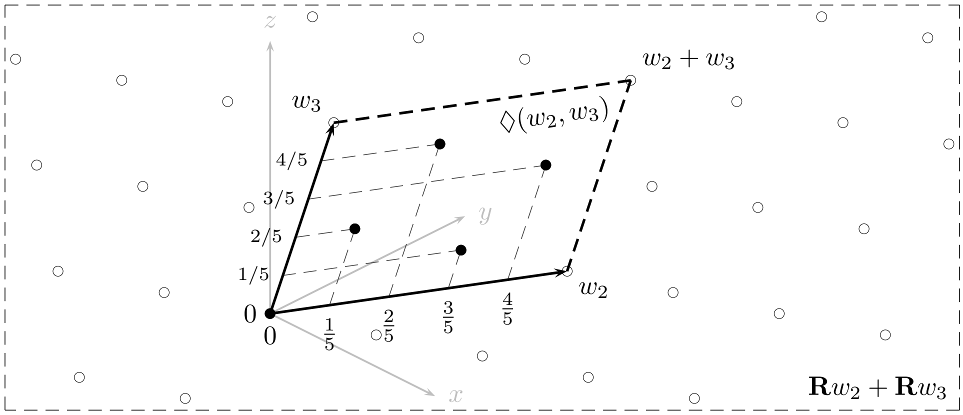


FIGURE 3. Example of a fundamental parallelepiped $\lozenge(w_2, w_3)$ spanned by two primitive vectors w_2 and w_3 in $\mathbf{Z}^3_{\geqslant 0}$. The dots represent the integral points in the plane $\mathbf{R}w_2 + \mathbf{R}w_3$; the solid dots are the integral points inside the fundamental parallelepiped and make up H_1. Observe the coordinates (h_2, h_3) of the points $h \in H_1$ with respect to the basis (w_2, w_3). In this example the multiplicity μ_1 of $\lozenge(w_2, w_3)$ equals 5 and $\xi_1 = 2$.

Since $h_2, h_3 \in (1/\mu_1)\mathbf{Z}$ and $0 \leqslant h_2, h_3 < 1$, the only possible values for h_2, h_3 are

$$0, \frac{1}{\mu_1}, \frac{2}{\mu_1}, \ldots, \frac{\mu_1 - 1}{\mu_1}.$$

Moreover, since w_2, w_3 are primitive, every i/μ_1, $i \in \{0, \ldots, \mu_1 - 1\}$, is the w_2-coordinate (w_3-coordinate) of at most, and therefore exactly, one point $h \in H_1$:

$$\{h_2 \mid h \in H_1\} = \{h_3 \mid h \in H_1\} = \left\{0, \frac{1}{\mu_1}, \frac{2}{\mu_1}, \ldots, \frac{\mu_1 - 1}{\mu_1}\right\}.$$

So there exists a unique $\xi_1 \in \{0, \ldots, \mu_1 - 1\}$ such that

$$h^* = \frac{1}{\mu_1} w_2 + \frac{\xi_1}{\mu_1} w_3 \in H_1. \tag{7}$$

Consider the cyclic subgroup $\langle h^* \rangle \subseteq H_1$ generated by h^*. This subgroup contains the μ_1 distinct elements

$$\{ih^*\} = \frac{i}{\mu_1} w_2 + \left\{\frac{i\xi_1}{\mu_1}\right\} w_3; \qquad i = 0, \ldots, \mu_1 - 1;$$

of H_1, and therefore equals H_1. Figure 3 illustrates the situation. This gives us a complete[18] description of the points of H_1. Besides, since $h_3 = \{i\xi_1/\mu_1\}$ runs through $\{0, 1/\mu_1, \ldots, (\mu_1 - 1)/\mu_1\}$ when i runs through $\{0, \ldots, \mu_1 - 1\}$, we have that $\xi_1 + \mu_1\mathbf{Z}$ generates $\mathbf{Z}/\mu_1\mathbf{Z}, +$ and therefore $\gcd(\xi_1, \mu_1) = 1$. Obviously, analogous results hold for H_2 and H_3, concluding Theorem 1.7(iii).

[18] In Paragraph 1.7.1 we explain how to obtain ξ_1 from the coordinates of w_2 and w_3.

1.4. **On the w_1-coordinates of the points of H.** Let $h \in H$ and write

$$h = h_1 w_1 + h_2 w_2 + h_3 w_3 \tag{8}$$

with $h_1, h_2, h_3 \in [0,1)$. Because $|H| = \mu$, it holds that

$$\{\mu h\} = \{\mu h_1\} w_1 + \{\mu h_2\} w_2 + \{\mu h_3\} w_3 = (0,0,0),$$

and therefore

$$h_1, h_2, h_3 \in \left\{0, \frac{1}{\mu}, \frac{2}{\mu}, \ldots, \frac{\mu - 1}{\mu}\right\}.$$

Let us study the w_1-coordinates of the μ points of H in more detail. Note that the μ/μ_1 cosets of the subgroup H_1 of H form the equivalence classes of the equivalence relation $\sim$ on H defined by

$$h \sim h' \qquad \text{if and only if} \qquad h_1 = h_1';$$

i.e., $H/H_1 = H/\sim$ as sets. This implies that there are μ/μ_1 possible values for the w_1-coordinate of a point of H, and since every coset of H_1 contains μ_1 elements, every possible w_1-coordinate occurs precisely μ_1 times.

Moreover, the classes modulo $\mathbf{Z}$ of the possible w_1-coordinates form a subgroup of $(1/\mu)\mathbf{Z}/\mathbf{Z}$, isomorphic to H/H_1. The possible values for the coordinates h_1 of the points $h \in H$ are therefore

$$0, \frac{\mu_1}{\mu}, \frac{2\mu_1}{\mu}, \ldots, \frac{\mu - \mu_1}{\mu},$$

and every $l\mu_1/\mu$, $l \in \{0, \ldots, \mu/\mu_1 - 1\}$, is the w_1-coordinate of exactly μ_1 points of H. Again, there are similar results for the other two coordinates h_2 and h_3. We conclude Theorem 1.7(iv).

1.5. **More divisibility relations.**

Notation 1.9. For the remaining of this section, we will use the following notations:

$$\gamma = \gcd(\mu_1, \mu_2) \qquad \text{and} \qquad \lambda = \operatorname{lcm}(\mu_1, \mu_2) = \frac{\mu_1 \mu_2}{\gamma}.$$

We will denote as well $\mu_1' = \mu_1/\gamma$ and $\mu_2' = \mu_2/\gamma$.

Recall that

$$\mu = \mu_1 \mu_2 \varphi_3 = \mu_1 \mu_3 \varphi_2 = \mu_2 \mu_3 \varphi_1.$$

It follows that $\mu_1\mu_3, \mu_2\mu_3 \mid \mu_1\mu_2\varphi_3$. Hence $\mu_3 \mid \mu_1\varphi_3, \mu_2\varphi_3$ and thus $\mu_3 \mid \gamma\varphi_3$.

We already know that the subgroup $H_1 + H_2$ of H is isomorphic to the direct sum $H_1 \oplus H_2$ of H_1 and H_2 and therefore contains $\mu_1\mu_2$ elements. We can write down the $\mu_1\mu_2$ points of $H_1 + H_2$ explicitly.

The μ_1 points of H_1 are given by

$$\left\{\frac{i\xi_1'}{\mu_1}\right\} w_2 + \frac{i}{\mu_1} w_3; \qquad i = 0, \ldots, \mu_1 - 1;$$

for some uniquely determined $\xi_1' \in \{0, \ldots, \mu_1 - 1\}$ with $\gcd(\xi_1', \mu_1) = 1$. We prefer this representation (with ξ_1' instead of ξ_1) of the points of H_1 to the one on p. 22 (Equation (7)), because this one is more convenient for what follows.

In the same way, we can list the points of H_2 as

$$\left\{\frac{j\xi_2'}{\mu_2}\right\} w_1 + \frac{j}{\mu_2} w_3; \qquad j = 0, \ldots, \mu_2 - 1;$$

for some uniquely determined $\xi_2' \in \{0, \ldots, \mu_2 - 1\}$ with $\gcd(\xi_2', \mu_2) = 1$. Consequently, $H_1 + H_2$ consists of the following $\mu_1\mu_2$ points:

$$\left\{\frac{j\xi_2'}{\mu_2}\right\} w_1 + \left\{\frac{i\xi_1'}{\mu_1}\right\} w_2 + \left\{\frac{j\mu_1 + i\mu_2}{\mu_1\mu_2}\right\} w_3; \qquad i = 0, \ldots, \mu_1 - 1; \quad j = 0, \ldots, \mu_2 - 1. \tag{9}$$

Let us take a look at the w_3-coordinates of the above points. We see that for each $h \in H_1 + H_2$, the w_3-coordinate h_3 is a multiple of $\gamma/\mu_1\mu_2$. Indeed, for all i, j it holds that $\gamma = \gcd(\mu_1, \mu_2) \mid j\mu_1 + i\mu_2$. Moreover, $\gamma/\mu_1\mu_2$ and all of its multiples in $[0, 1)$ are the w_3-coordinate of some point in $H_1 + H_2$. Indeed, if we write γ as $\gamma = \alpha_1\mu_1 + \alpha_2\mu_2$ with $\alpha_1, \alpha_2 \in \mathbf{Z}$, we have that

$$\frac{\gamma}{\mu_1\mu_2} = \left\{\frac{j\mu_1 + i\mu_2}{\mu_1\mu_2}\right\}$$

for $j = \{\alpha_1\}_{\mu_2}$ and $i = \{\alpha_2\}_{\mu_1}$. It follows that

$$\begin{aligned}
&\{h_3 \mid h \in H_1 + H_2\} \\
&\quad = \left\{ \left\{\frac{j\mu_1 + i\mu_2}{\mu_1\mu_2}\right\} \;\middle|\; i \in \{0, \ldots, \mu_1 - 1\},\ j \in \{0, \ldots, \mu_2 - 1\} \right\} \\
&\quad = \left\{0, \frac{\gamma}{\mu_1\mu_2}, \frac{2\gamma}{\mu_1\mu_2}, \ldots, \frac{\mu_1\mu_2 - \gamma}{\mu_1\mu_2}\right\} \\
&\quad = \left\{0, \frac{1}{\lambda}, \frac{2}{\lambda}, \ldots, \frac{\lambda - 1}{\lambda}\right\}.
\end{aligned}$$

As we know, every multiple of μ_3/μ in $[0, 1)$ is the w_3-coordinate of some point in H. Choose $h^* \in H$ with $h_3^* = \mu_3/\mu$, and consider the coset $h^* + (H_1 + H_2)$ of $H_1 + H_2$ in the quotient group $H/(H_1 + H_2)$. Since

$$\left|\frac{H}{H_1 + H_2}\right| = \frac{|H|}{|H_1 + H_2|} = \frac{\mu}{\mu_1\mu_2} = \varphi_3,$$

it holds that

$$\big\{\varphi_3\big(h^* + (H_1 + H_2)\big)\big\} = \{\varphi_3 h^*\} + (H_1 + H_2) = H_1 + H_2,$$

and thus $\{\varphi_3 h^*\} \in H_1 + H_2$.

The w_3-coordinate

$$\{\varphi_3 h_3^*\} = \left\{\frac{\varphi_3\mu_3}{\mu}\right\} = \left\{\frac{\mu_3}{\mu_1\mu_2}\right\}$$

of $\{\varphi_3 h^*\}$ therefore must equal

$$\left\{\frac{\mu_3}{\mu_1\mu_2}\right\} = \left\{\frac{j\mu_1 + i\mu_2}{\mu_1\mu_2}\right\}$$

for some $i \in \{0, \ldots, \mu_1 - 1\}$ and some $j \in \{0, \ldots, \mu_2 - 1\}$. It follows that μ_3 is a $\mathbf{Z}$-linear combination of μ_1 and μ_2; hence $\gamma \mid \mu_3$.

Next, we will count the number of points in $(H_1 + H_2) \cap H_3$. Based on the explicit description (9) of the $\mu_1\mu_2$ points of $H_1 + H_2$, we have to examine for which $(i, j) \in \{0, \ldots, \mu_1 - 1\} \times \{0, \ldots, \mu_2 - 1\}$ it holds that

$$h_3 = \left\{\frac{j\mu_1 + i\mu_2}{\mu_1\mu_2}\right\} = 0.$$

Since
$$0 \leqslant \frac{j\mu_1 + i\mu_2}{\mu_1\mu_2} < 2,$$
we have that $h_3 = 0$ if and only if $i = j = 0$ or
$$j\mu_1 + i\mu_2 = \mu_1\mu_2. \tag{10}$$
For this last equality to hold, it is necessary that $\mu_2 \mid j\mu_1$ and $\mu_1 \mid i\mu_2$, which is equivalent to[19] $\lambda \mid j\mu_1, i\mu_2$, and even to $\mu_2' \mid j$ and $\mu_1' \mid i$. In other words, Equality (10) implies that
$$j = \frac{g\mu_2}{\gamma} \qquad \text{and} \qquad i = \frac{g'\mu_1}{\gamma}$$
for certain $g, g' \in \{0, \ldots, \gamma - 1\}$ and is therefore equivalent to
$$j = \frac{g\mu_2}{\gamma} \qquad \text{and} \qquad i = \frac{(\gamma - g)\mu_1}{\gamma}$$
for some $g \in \{1, \ldots, \gamma - 1\}$.

We can conclude that $(H_1 + H_2) \cap H_3$ contains precisely $\gamma = \gcd(\mu_1, \mu_2)$ points, and they are given by
$$h = \left\{\frac{g\xi_2'}{\gamma}\right\} w_1 + \left\{\frac{(\gamma - g)\xi_1'}{\gamma}\right\} w_2; \qquad g = 0, \ldots, \gamma - 1.$$

Because $\xi_2' + \mu_2\mathbf{Z}$ and $\xi_1' + \mu_1\mathbf{Z}$ generate $\mathbf{Z}/\mu_2\mathbf{Z}$ and $\mathbf{Z}/\mu_1\mathbf{Z}$, respectively, and $\gamma \mid \mu_1, \mu_2$, it follows that $\xi_1' + \gamma\mathbf{Z}$ and $\xi_2' + \gamma\mathbf{Z}$ are both generators of $\mathbf{Z}/\gamma\mathbf{Z}$. (Moreover, the map $g \mapsto \{\gamma - g\}_\gamma$ is a permutation of $\{0, \ldots, \gamma - 1\}$.) The coordinates
$$h_1(g) = \left\{\frac{g\xi_2'}{\gamma}\right\} \qquad \text{and} \qquad h_2(g) = \left\{\frac{(\gamma - g)\xi_1'}{\gamma}\right\}$$
therefore both run through all the elements of
$$\left\{0, \frac{1}{\gamma}, \frac{2}{\gamma}, \ldots, \frac{\gamma - 1}{\gamma}\right\}$$
when g runs through $\{0, \ldots, \gamma - 1\}$. (Hence the maps $g \mapsto h_1(g)$ and $g \mapsto h_2(g)$ from $\{0, \ldots, \gamma - 1\}$ to $\{0, 1/\gamma, \ldots, (\gamma - 1)/\gamma\}$ are both bijections.)

This leads to the existence of a unique $\xi_\gamma \in \{0, \ldots, \gamma - 1\}$, coprime to γ, such that the elements of $(H_1 + H_2) \cap H_3$ can be represented as
$$\frac{g}{\gamma} w_1 + \left\{\frac{g\xi_\gamma}{\gamma}\right\} w_2; \qquad g = 0, \ldots, \gamma - 1. \tag{11}$$
Hence $(H_1 + H_2) \cap H_3$ is the cyclic subgroup of H generated by $(1/\gamma)w_1 + (\xi_\gamma/\gamma)w_2$.

Remark 1.10. The number ξ_γ appearing in (11) is determined by the equality
$$\xi_\gamma + \gamma\mathbf{Z} = -(\xi_2' + \gamma\mathbf{Z})^{-1}(\xi_1' + \gamma\mathbf{Z})$$
in the ring $\mathbf{Z}/\gamma\mathbf{Z}, +, \cdot$. Furthermore, we have that $\xi_\gamma = \{\xi_3\}_\gamma$, with ξ_3 the unique element of $\{0, \ldots, \mu_3 - 1\}$ such that
$$\frac{1}{\mu_3} w_1 + \frac{\xi_3}{\mu_3} w_2 \in H_3.$$

[19]Cfr. Notation 1.9.

Remark 1.11. From $\gamma = \gcd(\mu_1, \mu_2) \mid \mu_3$, we see that in fact

$$\gamma = \gcd(\mu_1, \mu_2, \mu_3) = \gcd(|d_{ij}|)_{i,j=1,2,3},$$

and by symmetry, $\gamma = \gcd(\mu_1, \mu_2) = \gcd(\mu_1, \mu_3) = \gcd(\mu_2, \mu_3)$.

The quotient group

$$\frac{H_1 + H_2}{(H_1 + H_2) \cap H_3} \tag{12}$$

of order $\mu_1\mu_2/\gamma = \lambda$ partitions $H_1 + H_2$ based on the w_3-coordinates of its points, and

$$\{h_3 + \mathbf{Z} \mid h \in H_1 + H_2\}, +$$

therefore is the unique subgroup of $\mathbf{R}/\mathbf{Z}, +$ of order λ. The set of w_3-coordinates of points of $H_1 + H_2$ is thus $\{0, 1/\lambda, \ldots, (\lambda - 1)/\lambda\}$ (we already knew that), and since every coset of $(H_1 + H_2) \cap H_3$ counts γ points, every l/λ, $l \in \{0, \ldots, \lambda - 1\}$, is the w_3-coordinate of precisely γ points in $H_1 + H_2$. This ends the proof of Theorem 1.7(v).

1.6. **Explicit description of the points of H.** We shall write the μ_1 points of H_1 and the μ_2 points of H_2 as

$$h(i, 0, 0) = \frac{i}{\mu_1} w_2 + \left\{ \frac{i\xi_1}{\mu_1} \right\} w_3; \qquad i = 0, \ldots, \mu_1 - 1;$$

and

$$h(0, j, 0) = \frac{j}{\mu_2} w_1 + \left\{ \frac{j\xi_2}{\mu_2} \right\} w_3; \qquad j = 0, \ldots, \mu_2 - 1;$$

respectively. The $\mu_1\mu_2$ points of $H_1 + H_2$ are then given by

$$h(i, j, 0) = \frac{j}{\mu_2} w_1 + \frac{i}{\mu_1} w_2 + \left\{ \frac{i\xi_1\mu_2 + j\xi_2\mu_1}{\mu_1\mu_2} \right\} w_3;$$
$$i = 0, \ldots, \mu_1 - 1; \quad j = 0, \ldots, \mu_2 - 1.$$

The w_3-coordinate $h_3(i, j, 0)$ of $h(i, j, 0)$ can also be written as

$$h_3(i, j, 0) = \left\{ \frac{i\xi_1\mu_2 + j\xi_2\mu_1}{\mu_1\mu_2} \right\} = \frac{l(i, j, 0)\gamma}{\mu_1\mu_2} = \frac{l(i, j, 0)}{\lambda},$$

with

$$l(i, j, 0) = \frac{\{i\xi_1\mu_2 + j\xi_2\mu_1\}_{\mu_1\mu_2}}{\gamma} = \left\{ \frac{i\xi_1\mu_2 + j\xi_2\mu_1}{\gamma} \right\}_\lambda$$
$$= \{i\xi_1\mu_2' + j\xi_2\mu_1'\}_\lambda \in \{0, \ldots, \lambda - 1\}$$

for all i, j. This results in

$$h(i, j, 0) = \frac{j}{\mu_2} w_1 + \frac{i}{\mu_1} w_2 + \frac{l(i, j, 0)}{\lambda} w_3; \qquad i = 0, \ldots, \mu_1 - 1; \quad j = 0, \ldots, \mu_2 - 1;$$

whereby $l(i, j, 0)$ runs exactly γ times through all the elements of $\{0, \ldots, \lambda - 1\}$ when i and j run through $\{0, \ldots, \mu_1 - 1\}$ and $\{0, \ldots, \mu_2 - 1\}$, respectively.

Because H is the disjoint union of the φ_3 cosets of $H_1 + H_2$ in H, we can describe all elements of H by choosing representatives $h(0, 0, k)$; $k = 0, \ldots, \varphi_3 - 1$; one for each coset, and then view H as the set

$$H = H_1 + H_2 + \{h(0, 0, 1), \ldots, h(0, 0, \varphi_3 - 1)\}.$$

We know that every $h \in H$ has a w_1-coordinate h_1 of the form $h_1 = t\mu_1/\mu$ for some $t \in \{0, \ldots, \mu/\mu_1 - 1\}$, i.e., of the form

$$h_1 = \frac{j\varphi_3 + k}{\mu_2\varphi_3}$$

for some $j \in \{0, \ldots, \mu_2 - 1\}$ and some $k \in \{0, \ldots, \varphi_3 - 1\}$, and that every such number $(j\varphi_3 + k)/\mu_2\varphi_3$ is the w_1-coordinate of precisely μ_1 points of H. In this way, we can associate to each $h \in H$ a number $k = \{\mu_2\varphi_3 h_1\}_{\varphi_3} \in \{0, \ldots, \varphi_3 - 1\}$, and we see that $H/(H_1 + H_2)$ is the partition of H based on these values of k. The analogous result holds for the w_2-coordinates of the points of H.

The w_3-coordinate h_3 of every point $h \in H$ has the form $h_3 = t\mu_3/\mu$ for some $t \in \{0, \ldots, \mu/\mu_3 - 1\}$, and every $t\mu_3/\mu$ is the w_3-coordinate of precisely μ_3 points of H. Since $\gamma \mid \mu_3 \mid \gamma\varphi_3$ and we actually have $\gamma = \gcd(\mu_1, \mu_2, \mu_3)$, we can put $\mu_3 = \gamma\mu_3'$ and $\varphi_3 = \mu_3'\varphi_3'$ with $\mu_3', \varphi_3' \in \mathbf{Z}_{>0}$.

We can now write every w_3-coordinate h_3 as

$$h_3 = \frac{t\mu_3}{\mu} = \frac{t\gamma\mu_3'}{\mu_1\mu_2\varphi_3} = \frac{t}{\lambda\varphi_3'}$$

for some $t \in \{0, \ldots, \lambda\varphi_3' - 1\}$, and thus as

$$h_3 = \frac{t}{\lambda\varphi_3'} = \frac{l\varphi_3' + k'}{\lambda\varphi_3'}$$

for some $l \in \{0, \ldots, \lambda - 1\}$ and some $k' \in \{0, \ldots, \varphi_3' - 1\}$.

The $k' = \{\lambda\varphi_3' h_3\}_{\varphi_3'}$, associated in this way to every $h \in H$, is constant on the cosets of $H_1 + H_2$, but points in different cosets may have the same value for k'. In fact each of the φ_3' possible values for k' is adopted in precisely μ_3' cosets of $H_1 + H_2$. (This agrees with the fact that every $(l\varphi_3' + k')/\lambda\varphi_3'$ appears precisely $\mu_3 = \gamma\mu_3'$ times as the w_3-coordinate of a point of H, considered that each l/λ is the w_3-coordinate of precisely γ points in $H_1 + H_2$.)

We can now choose representatives for the elements of $H/(H_1 + H_2)$. First, choose a point $h^* \in H$ with w_1-coordinate $h_1^* = 1/\mu_2\varphi_3$. The w_2-coordinate of this point equals

$$h_2^* = \frac{i_0\varphi_3 + \eta}{\mu_1\varphi_3}$$

for some $i_0 \in \{0, \ldots, \mu_1 - 1\}$ and some $\eta \in \{0, \ldots, \varphi_3 - 1\}$. All μ_1 points of H with w_1-coordinate $1/\mu_2\varphi_3$ are given by $\{h^* + h\}$, $h \in H_1$, and their w_2-coordinates by

$$\left\{\frac{(i_0 + i)\varphi_3 + \eta}{\mu_1\varphi_3}\right\}; \qquad i = 0, \ldots, \mu_1 - 1;$$

this is, after reordering, by

$$\frac{i\varphi_3 + \eta}{\mu_1\varphi_3}; \qquad i = 0, \ldots, \mu_1 - 1.$$

It follows that there exists a unique point $h(0,0,1) \in H$ of the form

$$h(0,0,1) = \frac{1}{\mu_2\varphi_3}w_1 + \frac{\eta}{\mu_1\varphi_3}w_2 + \frac{l_0\varphi_3' + \eta'}{\lambda\varphi_3'}w_3$$

with $\eta \in \{0, \ldots, \varphi_3 - 1\}$, $l_0 \in \{0, \ldots, \lambda - 1\}$, and $\eta' \in \{0, \ldots, \varphi_3' - 1\}$. We will choose this point $h(0,0,1)$ as the representative of its coset $h(0,0,1) + (H_1 + H_2)$.

The φ_3 multiples

$$\{kh(0,0,1)\} = \frac{k}{\mu_2\varphi_3}w_1 + \left\{\frac{k\eta}{\mu_1\varphi_3}\right\}w_2 + \left\{\frac{kl_0\varphi_3' + k\eta'}{\lambda\varphi_3'}\right\}w_3; \qquad k = 0,\ldots,\varphi_3 - 1;$$

of $h(0,0,1)$ run through all cosets of $H_1 + H_2$, and therefore, $h(0,0,1) + (H_1 + H_2)$ is a generator of the cyclic group $H/(H_1 + H_2)$.

We can choose the elements $\{kh(0,0,1)\}$; $k = 0,\ldots,\varphi_3 - 1$; as representatives of their respective cosets, but we can also choose, for each k, as a representative for $\{kh(0,0,1)\} + (H_1 + H_2)$, the unique element $h(0,0,k) \in \{kh(0,0,1)\} + H_1$ for which $h_2(0,0,k) < 1/\mu_1$. We take the last option. This is,

$$h(0,0,k) = \left\{kh(0,0,1) - \frac{i(k)}{\mu_1}w_2 - \frac{i(k)\xi_1}{\mu_1}w_3\right\},$$

with

$$i(k) = \left\{\frac{k\eta - \{k\eta\}_{\varphi_3}}{\varphi_3}\right\}_{\mu_1} = \left\{\left\lfloor\frac{k\eta}{\varphi_3}\right\rfloor\right\}_{\mu_1} \in \{0,\ldots,\mu_1 - 1\} \tag{13}$$

for $k = 0,\ldots,\varphi_3 - 1$, resulting in the following set of representatives for the elements of $H/(H_1 + H_2)$:

$$h(0,0,k) = \frac{k}{\mu_2\varphi_3}w_1 + \frac{\{k\eta\}_{\varphi_3}}{\mu_1\varphi_3}w_2 + \frac{l(k)\varphi_3' + \{k\eta'\}_{\varphi_3'}}{\lambda\varphi_3'}w_3; \qquad k = 0,\ldots,\varphi_3 - 1;$$

with for every k,

$$l(k) = \left\{kl_0 - i(k)\xi_1\mu_2' + \left\lfloor\frac{k\eta'}{\varphi_3'}\right\rfloor\right\}_\lambda \in \{0,\ldots,\lambda - 1\}, \tag{14}$$

$\mu_2' = \mu_2/\gamma$, and $i(k)$ as in (13).

When k runs through $\{0,\ldots,\varphi_3 - 1\}$, the coset $h(0,0,k) + (H_1 + H_2)$ runs through all elements of $H/(H_1 + H_2)$. This means that $\{k\eta\}_{\varphi_3}$ runs through $\{0,\ldots,\varphi_3 - 1\}$ once, while $\{k\eta'\}_{\varphi_3'}$ runs through $\{0,\ldots,\varphi_3' - 1\}$ precisely μ_3' times. It follows that $\eta + \varphi_3\mathbf{Z}$ and $\eta' + \varphi_3'\mathbf{Z}$ are generators of $\mathbf{Z}/\varphi_3\mathbf{Z}$ and $\mathbf{Z}/\varphi_3'\mathbf{Z}$, respectively, and therefore $\gcd(\eta,\varphi_3) = \gcd(\eta',\varphi_3') = 1$.

We can now list all the points of H. We start with an overview. The μ_1 points of H_1 are

$$h(i,0,0) = \frac{i}{\mu_1}w_2 + \left\{\frac{i\xi_1}{\mu_1}\right\}w_3; \qquad i = 0,\ldots,\mu_1 - 1;$$

while the μ_2 points of H_2 are given by

$$h(0,j,0) = \frac{j}{\mu_2}w_1 + \left\{\frac{j\xi_2}{\mu_2}\right\}w_3; \qquad j = 0,\ldots,\mu_2 - 1.$$

This gives the following $\mu_1\mu_2$ points for $H_1 + H_2$:

$$h(i,j,0) = \{h(i,0,0) + h(0,j,0)\} = \frac{j}{\mu_2}w_1 + \frac{i}{\mu_1}w_2 + \frac{l(i,j)}{\lambda}w_3; \qquad i = 0,\ldots,\mu_1 - 1; \quad j = 0,\ldots,\mu_2 - 1;$$

with for all i,j,

$$l(i,j) = \{i\xi_1\mu_2' + j\xi_2\mu_1'\}_\lambda \in \{0,\ldots,\lambda - 1\}.$$

As representatives for the φ_3 cosets of $H_1 + H_2$, we chose

$$h(0,0,k) = \frac{k}{\mu_2\varphi_3} w_1 + \frac{\{k\eta\}_{\varphi_3}}{\mu_1\varphi_3} w_2 + \frac{l(k)\varphi_3' + \{k\eta'\}_{\varphi_3'}}{\lambda\varphi_3'} w_3; \qquad k = 0, \ldots, \varphi_3 - 1;$$

with $l(k)$ as in (14).

Consequently, the $\mu = \mu_1\mu_2\varphi_3$ points of H are given by

$$\begin{aligned} h(i,j,k) &= \{h(i,0,0) + h(0,j,0) + h(0,0,k)\} \\ &= \frac{j\varphi_3 + k}{\mu_2\varphi_3} w_1 + \frac{i\varphi_3 + \{k\eta\}_{\varphi_3}}{\mu_1\varphi_3} w_2 + \frac{l(i,j,k)\varphi_3' + \{k\eta'\}_{\varphi_3'}}{\lambda\varphi_3'} w_3; \\ &\qquad i = 0, \ldots, \mu_1 - 1; \quad j = 0, \ldots, \mu_2 - 1; \quad k = 0, \ldots, \varphi_3 - 1; \end{aligned}$$

with for all i, j, k,

$$\begin{aligned} l(i,j,k) &= \{l(i,j) + l(k)\}_\lambda \\ &= \left\{ (i - i(k))\xi_1\mu_2' + j\xi_2\mu_1' + kl_0 + \left\lfloor \frac{k\eta'}{\varphi_3'} \right\rfloor \right\}_\lambda \in \{0, \ldots, \lambda - 1\} \qquad \text{and} \\ i(k) &= \left\{ \left\lfloor \frac{k\eta}{\varphi_3} \right\rfloor \right\}_{\mu_1} \in \{0, \ldots, \mu_1 - 1\}, \end{aligned}$$

and where

$$\begin{aligned} &\xi_1 \in \{0, \ldots, \mu_1 - 1\}, \qquad \eta \in \{0, \ldots, \varphi_3 - 1\}, \qquad l_0 \in \{0, \ldots, \lambda - 1\} \\ &\xi_2 \in \{0, \ldots, \mu_2 - 1\}, \qquad \eta' \in \{0, \ldots, \varphi_3' - 1\}, \end{aligned}$$

are uniquely determined by

$$\frac{1}{\mu_1} w_2 + \frac{\xi_1}{\mu_1} w_3, \quad \frac{1}{\mu_2} w_1 + \frac{\xi_2}{\mu_2} w_3, \quad \frac{1}{\mu_2\varphi_3} w_1 + \frac{\eta}{\mu_1\varphi_3} w_2 + \frac{l_0\varphi_3' + \eta'}{\lambda\varphi_3'} w_3 \quad \in H.$$

We repeat that when k runs through $\{0, \ldots, \varphi_3 - 1\}$, the numbers $\{k\eta\}_{\varphi_3}$ and $\{k\eta'\}_{\varphi_3'}$ run through $\{0, \ldots, \varphi_3 - 1\}$ and $\{0, \ldots, \varphi_3' - 1\}$ once and μ_3' times, respectively, while for fixed k, we have that $l(i,j,k)$ runs γ times through $\{0, \ldots, \lambda - 1\}$ when i and j run through $\{0, \ldots, \mu_1 - 1\}$ and $\{0, \ldots, \mu_2 - 1\}$, respectively. This concludes Theorem 1.7(vi).

1.7. Determination of the numbers $\xi_1, \xi_2, \xi_3, \eta, \eta', l_0$ from the coordinates of w_1, w_2, w_3.

1.7.1. *The numbers* ξ_1, ξ_2, ξ_3. We will give the explanation for ξ_1. Recall that we introduced ξ_1 as the unique element of $\{0, \ldots, \mu_1 - 1\}$ for which the μ_1 points of $H_1 = \mathbf{Z}^3 \cap \Diamond(w_2, w_3)$ are given by

$$\frac{i}{\mu_1} w_2 + \left\{ \frac{i\xi_1}{\mu_1} \right\} w_3; \qquad i = 0, \ldots, \mu_1 - 1.$$

How can we find ξ_1 from the coordinates of $w_2(a_2, b_2, c_2)$ and $w_3(a_3, b_3, c_3)$?

Consider the vector

$$\begin{aligned} -a_3w_2 + a_2w_3 &= (-a_3a_2 + a_2a_3, -a_3b_2 + a_2b_3, -a_3c_2 + a_2c_3) \\ &= (0, d_{13}, d_{12}) \in \mathbf{Z}^3. \end{aligned}$$

Since $\mu_1 = \gcd(|d_{11}|, |d_{12}|, |d_{13}|)$ divides every coordinate[20] of $-a_3w_2 + a_2w_3$, it holds that

$$\frac{1}{\mu_1}(-a_3w_2 + a_2w_3) = \frac{-a_3}{\mu_1}w_2 + \frac{a_2}{\mu_1}w_3 \in \mathbf{Z}^3.$$

On the other hand, we have

$$\frac{1}{\mu_1}w_2 + \frac{\xi_1}{\mu_1}w_3 \in \mathbf{Z}^3.$$

It follows that also

$$\frac{-a_3}{\mu_1}w_2 + \frac{-a_3\xi_1}{\mu_1}w_3 \in \mathbf{Z}^3 \qquad \text{and} \qquad \frac{a_2 + a_3\xi_1}{\mu_1}w_3 \in \mathbf{Z}^3.$$

Since w_3 is primitive, we obtain

$$a_3\xi_1 \equiv -a_2 \mod \mu_1.$$

Analogously, we find that

$$b_3\xi_1 \equiv -b_2 \mod \mu_1 \qquad \text{and} \qquad c_3\xi_1 \equiv -c_2 \mod \mu_1.$$

Consequently, ξ_1 is a solution of the following system of linear congruences:

$$\begin{cases} a_3x \equiv -a_2 \mod \mu_1, \\ b_3x \equiv -b_2 \mod \mu_1, \\ c_3x \equiv -c_2 \mod \mu_1. \end{cases} \tag{15}$$

The first linear congruence has a solution if and only if $\gcd(a_3, \mu_1) \mid a_2$. We show that this is indeed the case. Put

$$\gamma_a = \gcd(a_3, \mu_1) = \gcd(a_3, |d_{11}|, |d_{12}|, |d_{13}|).$$

It then follows from $\gamma_a \mid a_3, d_{12}, d_{13}$ that $\gamma_a \mid a_2a_3, a_2b_3, a_2c_3$. Hence

$$\gamma_a \mid a_2 \gcd(a_3, b_3, c_3) = a_2.$$

Analogously we have $\gcd(a_2, \mu_1) \mid a_3$, and thus we can write

$$\gamma_a = \gcd(a_2, \mu_1) = \gcd(a_3, \mu_1).$$

In the same way the other two congruences have solutions, and we may put

$$\gamma_b = \gcd(b_2, \mu_1) = \gcd(b_3, \mu_1) \qquad \text{and}$$
$$\gamma_c = \gcd(c_2, \mu_1) = \gcd(c_3, \mu_1).$$

The system (15) is then equivalent to

$$\begin{cases} x \equiv -a_2'\{a_3'\}^{-1}_{\mu_1^{(a)}} \mod \mu_1^{(a)}, \\ x \equiv -b_2'\{b_3'\}^{-1}_{\mu_1^{(b)}} \mod \mu_1^{(b)}, \\ x \equiv -c_2'\{c_3'\}^{-1}_{\mu_1^{(c)}} \mod \mu_1^{(c)}, \end{cases} \tag{16}$$

with

$$a_2' = a_2/\gamma_a, \qquad a_3' = a_3/\gamma_a, \qquad \mu_1^{(a)} = \mu_1/\gamma_a,$$

and where $\{a_3'\}^{-1}_{\mu_1^{(a)}}$ denotes the unique element of $\{0, \ldots, \mu_1^{(a)} - 1\}$ such that

$$a_3'\{a_3'\}^{-1}_{\mu_1^{(a)}} \equiv 1 \mod \mu_1^{(a)}.$$

[20] Here we mean coordinates with respect to the standard basis of $\mathbf{R}^3$.

(Analogously for the numbers appearing in the other two congruences.)

Since the moduli $\mu_1^{(a)}, \mu_1^{(b)}, \mu_1^{(c)}$ are generally not pairwise coprime, according to the Generalized Chinese Remainder Theorem, the system (16) has a solvability condition in the form of

$$a_2'\{a_3'\}^{-1}_{\mu_1^{(a)}} \equiv b_2'\{b_3'\}^{-1}_{\mu_1^{(b)}} \mod \gcd(\mu_1^{(a)}, \mu_1^{(b)}), \tag{17}$$

together with the analogous conditions for the other two combinations of two out of three congruences. Of course we know that the system is solvable since ξ_1 is a solution, but for the sake of completeness, let us verify Condition (17) in a direct way.

Because a_3' and b_3' are units modulo $\mu_1^{(a)}$ and $\mu_1^{(b)}$, respectively, they are both units modulo $\gcd(\mu_1^{(a)}, \mu_1^{(b)})$. Furthermore, we have that

$$a_3'\{a_3'\}^{-1}_{\mu_1^{(a)}} \equiv b_3'\{b_3'\}^{-1}_{\mu_1^{(b)}} \equiv 1 \mod \gcd\big(\mu_1^{(a)}, \mu_1^{(b)}\big). \tag{18}$$

If we multiply both sides of (17) with the unit $a_3'b_3'$ and apply (18), we find that Condition (17) is equivalent to

$$a_2'b_3' \equiv a_3'b_2' \mod \gcd(\mu_1^{(a)}, \mu_1^{(b)}),$$

and—after multiplying both sides and the modulus with $\gamma_a\gamma_b$—even to

$$\begin{aligned}\gcd(a_2, a_3, b_2, b_3)\mu_1 &= \gcd(\gamma_a, \gamma_b)\mu_1\\ &= \gamma_a\gamma_b \gcd(\mu_1^{(a)}, \mu_1^{(b)}) \mid \gamma_a\gamma_b(a_2'b_3' - a_3'b_2') = d_{13}.\end{aligned}$$

Of course we have that $\mu_1 \mid d_{13}$. It is therefore sufficient to show that for every prime p with $p \mid \gcd(a_2, a_3, b_2, b_3)$, it holds that

$$\operatorname{ord}_p d_{13} \geqslant \operatorname{ord}_p \gcd(a_2, a_3, b_2, b_3) + \operatorname{ord}_p \mu_1.$$

Let p be such a prime. Since $p \mid a_2, b_2$ and w_2 is primitive, it certainly holds that $p \nmid c_2$. It now follows from $a_2d_{11} - b_2d_{12} + c_2d_{13} = 0$ that

$$\begin{aligned}\operatorname{ord}_p d_{13} &= \operatorname{ord}_p c_2d_{13}\\ &= \operatorname{ord}_p(-a_2d_{11} + b_2d_{12})\\ &\geqslant \min\{\operatorname{ord}_p a_2 + \operatorname{ord}_p d_{11}, \operatorname{ord}_p b_2 + \operatorname{ord}_p d_{12}\}\\ &\geqslant \min\{\operatorname{ord}_p a_2, \operatorname{ord}_p a_3, \operatorname{ord}_p b_2, \operatorname{ord}_p b_3\}\\ &\qquad\qquad + \min\{\operatorname{ord}_p d_{11}, \operatorname{ord}_p d_{12}, \operatorname{ord}_p d_{13}\}\\ &= \operatorname{ord}_p \gcd(a_2, a_3, b_2, b_3) + \operatorname{ord}_p \mu_1.\end{aligned}$$

The system is thus indeed solvable and the Generalized Chinese Remainder Theorem asserts that its solution is unique modulo

$$\operatorname{lcm}(\mu_1^{(a)}, \mu_1^{(b)}, \mu_1^{(c)}) = \mu_1.$$

We can thus find ξ_1 as the unique solution in $\{0, \ldots, \mu_1 - 1\}$ of the system (15).

1.7.2. *Determination of* η, η', *and* l_0. Recall that we introduced the numbers η, η', l_0 as the unique $\eta \in \{0, \ldots, \varphi_3 - 1\}$, $l_0 \in \{0, \ldots, \lambda - 1\}$, and $\eta' \in \{0, \ldots, \varphi_3' - 1\}$ such that

$$\frac{1}{\mu_2\varphi_3} w_1 + \frac{\eta}{\mu_1\varphi_3} w_2 + \frac{l_0\varphi_3' + \eta'}{\lambda\varphi_3'} w_3 \in H. \tag{19}$$

Recall as well that $(\operatorname{adj} M)M = dI$, with $d = \det M$; i.e.,

$$\begin{pmatrix} d_{11} & -d_{21} & d_{31} \\ -d_{12} & d_{22} & -d_{32} \\ d_{13} & -d_{23} & d_{33} \end{pmatrix} \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = \begin{pmatrix} d & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{pmatrix}. \tag{20}$$

Let $j \in \{1, 2, 3\}$. Since $\mu = |d|$ divides d, it follows from (20) that

$$h(j) = \frac{d_{1j}}{\mu} w_1 - \frac{d_{2j}}{\mu} w_2 + \frac{d_{3j}}{\mu} w_3 \in \mathbf{Z}^3.$$

Recall also that $\mu = \mu_1\mu_2\varphi_3$ and that

$$\mu_i = \gcd(|d_{i1}|, |d_{i2}|, |d_{i3}|); \qquad i = 1, 2, 3.$$

If we now put $d'_{ij} = d_{ij}/\mu_i$; $i = 1, 2, 3$; we obtain

$$\begin{aligned} h(j) &= \frac{d'_{1j}}{\mu_2\varphi_3} w_1 - \frac{d'_{2j}}{\mu_1\varphi_3} w_2 + \frac{d'_{3j}\mu_3}{\mu_1\mu_2\varphi_3} w_3 \\ &= \frac{d'_{1j}}{\mu_2\varphi_3} w_1 - \frac{d'_{2j}}{\mu_1\varphi_3} w_2 + \frac{d'_{3j}}{\lambda\varphi_3'} w_3 \in \mathbf{Z}^3. \end{aligned}$$

On the other hand, we also know the point

$$h'(j) = \frac{d'_{1j}}{\mu_2\varphi_3} w_1 + \frac{d'_{1j}\eta}{\mu_1\varphi_3} w_2 + \frac{d'_{1j}(l_0\varphi_3' + \eta')}{\lambda\varphi_3'} w_3 \in \mathbf{Z}^3$$

with the same w_1-coordinate as $h(j)$. After reduction of its coordinates[21] modulo one, $h(j) - h'(j)$ thus belongs to H_1, and since the coordinates[21] of the elements of H_1 belong to $(1/\mu_1)\mathbf{Z}$, we have that

$$\begin{aligned} \frac{d'_{1j}\eta}{\mu_1\varphi_3} &\equiv -\frac{d'_{2j}}{\mu_1\varphi_3} \mod \frac{1}{\mu_1} \qquad \text{and} \\ \frac{d'_{1j}(l_0\varphi_3' + \eta')}{\lambda\varphi_3'} &\equiv \frac{d'_{3j}}{\lambda\varphi_3'} \mod \frac{1}{\mu_1}, \end{aligned}$$

or, equivalently, that

$$\begin{aligned} d'_{1j}\eta &\equiv -d'_{2j} \mod \varphi_3 \qquad \text{and} \\ d'_{1j}\eta' &\equiv d'_{3j} \mod \mu_2'\varphi_3'. \end{aligned}$$

(Recall that $\lambda = \operatorname{lcm}(\mu_1, \mu_2) = \mu_1\mu_2/\gamma = \mu_1\mu_2'$.) A fortiori, it thus holds that

$$d'_{1j}\eta' \equiv d'_{3j} \mod \varphi_3'.$$

We have just showed that η and η' are solutions of the respective systems of linear congruences

$$\begin{cases} d'_{11}x \equiv -d'_{21} \mod \varphi_3, \\ d'_{12}x \equiv -d'_{22} \mod \varphi_3, \\ d'_{13}x \equiv -d'_{23} \mod \varphi_3; \end{cases} \qquad \text{and} \qquad \begin{cases} d'_{11}x \equiv d'_{31} \mod \varphi_3', \\ d'_{12}x \equiv d'_{32} \mod \varphi_3', \\ d'_{13}x \equiv d'_{33} \mod \varphi_3'. \end{cases} \tag{21}$$

[21] Here we mean coordinates with respect to the basis (w_1, w_2, w_3).

Moreover, it turns out that η and η' are the unique solutions of these systems in $\{0,\ldots,\varphi_3-1\}$ and $\{0,\ldots,\varphi'_3-1\}$, respectively. This gives us a method to determine η and η' from the coordinates of w_1,w_2,w_3. We will study the first system of (21) in more detail, for the second system analogous conclusions will be true.

Let us verify the solvability conditions of the first system. The first linear congruence has solutions if and only if $\gcd(d'_{11},\varphi_3)\mid d'_{21}$, i.e., if and only if

$$\gcd(\mu_2 d_{11},\mu)\mid \mu_1 d_{21}.$$

Put $\Gamma_1=\gcd(\mu_2 d_{11},\mu)$. Then we have $\Gamma_1\mid\mu$ and $\Gamma_1\mid d_{11}d_{2j}$ for every $j\in\{1,2,3\}$, and it is sufficient to prove that $\Gamma_1\mid d_{1j}d_{21}$ for every j.

We already know that $\Gamma_1\mid d_{11}d_{21}$. Furthermore, from $\operatorname{adj}(\operatorname{adj}M)=dM$, it follows that

$$\begin{aligned} d_{11}d_{22}-d_{12}d_{21}&=\big(\operatorname{adj}(\operatorname{adj}M)\big)_{33}=dc_3 \qquad \text{and}\\ d_{11}d_{23}-d_{13}d_{21}&=\big(\operatorname{adj}(\operatorname{adj}M)\big)_{32}=db_3. \end{aligned} \tag{22}$$

We find thus that $\Gamma_1\mid\mu\mid dc_3=d_{11}d_{22}-d_{12}d_{21}$, and together with $\Gamma_1\mid d_{11}d_{22}$, this implies that $\Gamma_1\mid d_{12}d_{21}$. Analogously, it follows from (22) that $\Gamma_1\mid d_{13}d_{21}$. The first linear congruence therefore has solutions, and the same thing holds for the other two congruences.

The first system of (21) is now solvable if and only if for all $j_1,j_2\in\{1,2,3\}$, it holds that

$$d'_{1j_1}d'_{2j_2}\equiv d'_{1j_2}d'_{2j_1}\mod \gcd\left(\frac{\varphi_3}{\gamma_{j_1}},\frac{\varphi_3}{\gamma_{j_2}}\right), \tag{23}$$

with $\gamma_j=\gcd(d'_{1j},\varphi_3)$ for all j. Let us verify this for $j_1=1$ and $j_2=2$. For these values of j_1 and j_2, Condition (23) is equivalent to

$$d_{11}d_{22}\equiv d_{12}d_{21}\mod \gcd\left(\frac{\mu}{\gamma_1},\frac{\mu}{\gamma_2}\right),$$

which follows from $\mu\mid dc_3=d_{11}d_{22}-d_{12}d_{21}$.

The (Generalized) Chinese Remainder Theorem now states that the system has a unique solution modulo

$$\operatorname{lcm}\left(\frac{\varphi_3}{\gamma_1},\frac{\varphi_3}{\gamma_2},\frac{\varphi_3}{\gamma_3}\right)=\frac{\varphi_3}{\gcd(d'_{11},d'_{12},d'_{13},\varphi_3)}=\varphi_3.$$

An alternative way to find η and η', and a way to find l_0 is as follows. We know that

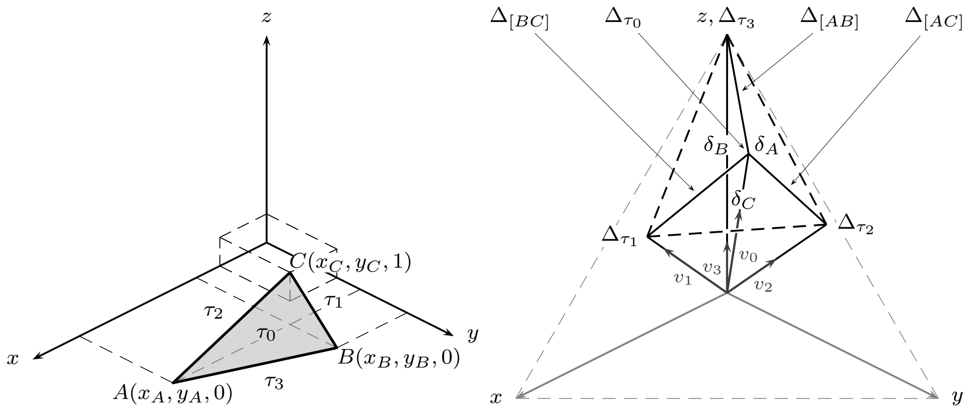
$$h(j)=\frac{d'_{1j}}{\mu_2\varphi_3}w_1-\frac{d'_{2j}}{\mu_1\varphi_3}w_2+\frac{d'_{3j}}{\lambda\varphi'_3}w_3\in\mathbf{Z}^3;\qquad j=1,2,3;$$

and that $\gcd(d'_{11},d'_{12},d'_{13})=1$. Find $\lambda_j\in\mathbf{Z}$; $j=1,2,3$; such that $\sum_j\lambda_j d'_{1j}=1$, and consider the point

$$\left\{\sum\nolimits_j\lambda_j h(j)\right\}=\frac{1}{\mu_2\varphi_3}w_1+\left\{\frac{-\sum_j\lambda_j d'_{2j}}{\mu_1\varphi_3}\right\}w_2+\left\{\frac{\sum_j\lambda_j d'_{3j}}{\lambda\varphi'_3}\right\}w_3\in H.$$

Substract from $\left\{\sum_j\lambda_j h(j)\right\}$ the point

$$\frac{i}{\mu_1}w_2+\left\{\frac{i\xi_1}{\mu_1}\right\}w_3\in H_1,$$



(a) B_1-simplex τ_0, its subfaces, and its neighbor facets τ_1, τ_2, and τ_3

(b) Relevant cones associated to relevant faces of Γ_f

FIGURE 4. Case I: the only facet contributing to s_0 is the B_1-simplex τ_0

with

$$i = \left\{ \left\lfloor \frac{-\sum_j \lambda_j d'_{2j}}{\varphi_3} \right\rfloor \right\}_{\mu_1},$$

and find the point

$$\frac{1}{\mu_2 \varphi_3} w_1 + \frac{\left\{-\sum_j \lambda_j d'_{2j}\right\}_{\varphi_3}}{\mu_1 \varphi_3} w_2 + \left\{ \frac{\sum_j \lambda_j d'_{3j} - i\xi_1 \mu'_2 \varphi'_3}{\lambda \varphi'_3} \right\} w_3 \in H.$$

Because of the uniqueness in H of a point of the form (19), we find that

$$\begin{aligned}
\eta &= \left\{ -\sum\nolimits_j \lambda_j d'_{2j} \right\}_{\varphi_3}, \\
\eta' &= \left\{ \sum\nolimits_j \lambda_j d'_{3j} \right\}_{\varphi'_3}, \qquad \text{and} \\
l_0 &= \left\{ \left\lfloor \frac{\sum_j \lambda_j d'_{3j}}{\varphi'_3} \right\rfloor - i\xi_1 \mu'_2 \right\}_{\lambda}.
\end{aligned}$$

2. CASE I: EXACTLY ONE FACET CONTRIBUTES TO s_0 AND THIS FACET IS A B_1-SIMPLEX

2.1. **Figure and notations.** Without loss of generality, we may assume that the B_1-simplex τ_0 contributing to s_0 is as drawn in Figure 4.

Let us fix notations. We shall denote the vertices of τ_0 and their coordinates by

$$A(x_A, y_A, 0), \quad B(x_B, y_B, 0), \quad \text{and} \quad C(x_C, y_C, 1).$$

The neighbor facets of τ_0 will be denoted τ_1, τ_2, τ_3, as indicated in Figure 4, and the unique primitive vectors perpendicular to them will be denoted by

$$v_0(a_0, b_0, c_0), \quad v_1(a_1, b_1, c_1), \quad v_2(a_2, b_2, c_2), \quad v_3(0, 0, 1),$$

respectively. Consequently, the affine supports of the considered facets should have equations of the form

$$\begin{aligned}
\operatorname{aff}(\tau_0) &\leftrightarrow a_0x + b_0y + c_0z = m_0,\\
\operatorname{aff}(\tau_1) &\leftrightarrow a_1x + b_1y + c_1z = m_1,\\
\operatorname{aff}(\tau_2) &\leftrightarrow a_2x + b_2y + c_2z = m_2,\\
\operatorname{aff}(\tau_3) &\leftrightarrow \qquad\qquad\qquad z = 0,
\end{aligned}$$

and we associate to them the following numerical data:

$$\begin{aligned}
m_0 &= m(v_0) = a_0x_C + b_0y_C + c_0, & \sigma_0 &= \sigma(v_0) = a_0 + b_0 + c_0,\\
m_1 &= m(v_1) = a_1x_C + b_1y_C + c_1, & \sigma_1 &= \sigma(v_1) = a_1 + b_1 + c_1,\\
m_2 &= m(v_2) = a_2x_C + b_2y_C + c_2, & \sigma_2 &= \sigma(v_2) = a_2 + b_2 + c_2,\\
m_3 &= m(v_3) = 0, & \sigma_3 &= \sigma(v_3) = 1.
\end{aligned}$$

We assume that τ_0 (and only τ_0) contributes to the candidate pole s_0. With the notations above this is, we assume that $p^{\sigma_0+m_0s_0} = 1$, or equivalently, that

$$\Re(s_0) = -\frac{\sigma_0}{m_0} = -\frac{a_0 + b_0 + c_0}{a_0x_C + b_0y_C + c_0} \qquad \text{and} \qquad \Im(s_0) = \frac{2n\pi}{m_0 \log p}$$

for some $n \in \mathbf{Z}$.

In this section we will consider the following simplicial cones:

$$\begin{aligned}
\delta_A &= \operatorname{cone}(v_0, v_2, v_3), & \Delta_{[AB]} &= \operatorname{cone}(v_0, v_3), & \Delta_{\tau_0} &= \operatorname{cone}(v_0).\\
\delta_B &= \operatorname{cone}(v_0, v_1, v_3), & \Delta_{[AC]} &= \operatorname{cone}(v_0, v_2),\\
\delta_C &= \operatorname{cone}(v_0, v_1, v_2), & \Delta_{[BC]} &= \operatorname{cone}(v_0, v_1),
\end{aligned}$$

The Δ_τ are the simplicial cones associated to the faces τ in the usual way. The cones $\Delta_A, \Delta_B, \Delta_C$, associated to the vertices of τ_0, are generally not simplicial. Later in this section we will consider simplicial subdivisions (without creating new rays) of $\Delta_A, \Delta_B, \Delta_C$ that contain the respective simplicial cones $\delta_A, \delta_B, \delta_C$.

Lastly, we fix notations for the vectors along the edges of τ_0:

$$\begin{aligned}
&\overrightarrow{AC}(x_C - x_A, y_C - y_A, 1) = (\alpha_A, \beta_A, 1),\\
&\overrightarrow{BC}(x_C - x_B, y_C - y_B, 1) = (\alpha_B, \beta_B, 1),\\
&\overrightarrow{AB}(x_B - x_A, y_B - y_A, 0) = (\alpha_A - \alpha_B, \beta_A - \beta_B, 0).
\end{aligned}$$

The first two vectors are primitive; the last one is generally not. We put

$$\varphi_{AB} = \gcd(x_B - x_A, y_B - y_A) = \gcd(\alpha_A - \alpha_B, \beta_A - \beta_B).$$

2.2. **Some relations between the variables.** Expressing that $\overrightarrow{AC} \perp v_0, v_2$ and $\overrightarrow{BC} \perp v_0, v_1$, we obtain

$$\begin{aligned}
\begin{pmatrix} c_0 \\ c_2 \end{pmatrix} &= -\alpha_A \begin{pmatrix} a_0 \\ a_2 \end{pmatrix} - \beta_A \begin{pmatrix} b_0 \\ b_2 \end{pmatrix} \qquad \text{and}\\
\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} &= -\alpha_B \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} - \beta_B \begin{pmatrix} b_0 \\ b_1 \end{pmatrix}.
\end{aligned}$$

These relations imply that

$$\gcd(a_i, b_i, c_i) = \gcd(a_i, b_i) = 1; \qquad i = 0, \ldots, 2.$$

Another consequence is that

$$\begin{vmatrix} a_0 & c_0 \\ a_2 & c_2 \end{vmatrix} = \begin{vmatrix} a_0 & -\alpha_A a_0 - \beta_A b_0 \\ a_2 & -\alpha_A a_2 - \beta_A b_2 \end{vmatrix} = -\beta_A \begin{vmatrix} a_0 & b_0 \\ a_2 & b_2 \end{vmatrix},$$

and analogously,

$$\begin{vmatrix} b_0 & c_0 \\ b_2 & c_2 \end{vmatrix} = \alpha_A \begin{vmatrix} a_0 & b_0 \\ a_2 & b_2 \end{vmatrix}, \quad \begin{vmatrix} a_0 & c_0 \\ a_1 & c_1 \end{vmatrix} = -\beta_B \begin{vmatrix} a_0 & b_0 \\ a_1 & b_1 \end{vmatrix}, \quad \begin{vmatrix} b_0 & c_0 \\ b_1 & c_1 \end{vmatrix} = \alpha_B \begin{vmatrix} a_0 & b_0 \\ a_1 & b_1 \end{vmatrix}.$$

It will turn out to be convenient (and sometimes necessary) to know the signs of certain determinants. Considering the orientations of the corresponding coordinate systems, one can show that

$$\begin{vmatrix} a_0 & b_0 \\ a_2 & b_2 \end{vmatrix} > 0, \quad \begin{vmatrix} a_0 & b_0 \\ a_1 & b_1 \end{vmatrix} < 0, \quad \Psi = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} > 0, \quad \text{and} \quad \begin{vmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} > 0.$$

2.3. **Igusa's local zeta function.** As f is non-degenerate over $\mathbf{F}_p$ with respect to the compact faces of its Newton polyhedron Γ_f, by Theorem 0.27 the local Igusa zeta function Z_f^0 of f is the meromorphic complex function

$$Z_f^0 = \sum_{\substack{\tau \text{ compact} \\ \text{face of } \Gamma_f}} L_\tau S(\Delta_\tau), \tag{24}$$

with

$$L_\tau : s \mapsto L_\tau(s) = \left(\frac{p-1}{p}\right)^3 - \frac{N_\tau}{p^2}\frac{p^s-1}{p^{s+1}-1},$$
$$N_\tau = \#\left\{(x,y,z) \in (\mathbf{F}_p^\times)^3 \;\middle|\; \overline{f_\tau}(x,y,z) = 0\right\},$$

and

$$S(\Delta_\tau) : s \mapsto S(\Delta_\tau)(s) = \sum_{k \in \mathbf{Z}^3 \cap \Delta_\tau} p^{-\sigma(k) - m(k)s}$$
$$= \sum_{i \in I} \frac{\Sigma(\delta_i)(s)}{\prod_{j \in J_i}(p^{\sigma(w_j)+m(w_j)s} - 1)}. \tag{25}$$

Here $\{\delta_i\}_{i \in I}$ denotes a simplicial decomposition without introducing new rays of the cone Δ_τ associated to τ. The simplicial cone δ_i is supposed to be strictly positively spanned by the linearly independent primitive vectors w_j, $j \in J_i$, in $\mathbf{Z}^n_{\geqslant 0} \setminus \{0\}$, and $\Sigma(\delta_i)$ is the function

$$\Sigma(\delta_i) : s \mapsto \Sigma(\delta_i)(s) = \sum_h p^{\sigma(h)+m(h)s},$$

where h runs through the elements of the set

$$H(w_j)_{j \in J_i} = \mathbf{Z}^3 \cap \Diamond(w_j)_{j \in J_i},$$

with

$$\Diamond(w_j)_{j \in J_i} = \left\{\sum\nolimits_{j \in J_i} h_j w_j \;\middle|\; h_j \in [0,1) \text{ for all } j \in J_i\right\}$$

the fundamental parallelepiped spanned by the vectors w_j, $j \in J_i$.

Remark 2.1. The formula above is generally valid, not particularly in Case I. We will use this formula throughout the entire proof (Sections 2–8).

2.4. **The candidate pole s_0 and its residue.**

Remark 2.2. This subsection is general to all cases where s_0 is a candidate pole of expected order one, i.e., to this case and the next.

We want to prove that s_0 is not a pole of Z_f^0. Since s_0 is a candidate pole of expected order one (and therefore is either no pole or a pole of order one), it is enough to prove that the coefficient a_{-1} in the Laurent series

$$Z_f^0(s) = \sum_{k=-1}^{\infty} a_k (s - s_0)^k$$

of Z_f^0 centered at s_0, equals zero. This coefficient, also called the residue of Z_f^0 in s_0, is given by

$$a_{-1} = \operatorname{Res}(Z_f^0, s_0) = \lim_{s \to s_0} (s - s_0) Z_f^0(s).$$

Equivalently, we will prove in the rest of this section that $R_1 = 0$, with

$$R_1 = \lim_{s \to s_0} \left(p^{\sigma_0 + m_0 s} - 1\right) Z_f^0(s) = (\log p) m_0 a_{-1}.$$

2.5. **Terms contributing to R_1.** We will next calculate R_1 based on Formula (24) for Z_f^0.

The only (compact) faces of Γ_f that contribute to the candidate pole s_0 are the subfaces $A, B, C, [AB], [AC], [BC], \tau_0$ of the single facet having s_0 as an associated candidate pole. They are only the terms of (24) corresponding to these faces that should be taken into account in the calculation of R_1:

$$R_1 = \lim_{s \to s_0} \left(p^{\sigma_0 + m_0 s} - 1\right) \sum_{\substack{\tau = \tau_0, A, B, C, \\ [AB], [AC], [BC]}} L_\tau(s) S(\Delta_\tau)(s).$$

A second simplification is the following. First, note that vertex A is contained in facets τ_0, τ_2, τ_3, but can still be contained in other facets. Hence Δ_A is—in general—not simplicial and the same thing holds for the other vertices B, C and their associated cones. Consequently, to handle S_A, S_B, and S_C, we need to consider simplicial decompositions of Δ_A, Δ_B, and Δ_C, and we will choose ones that contain the simplicial cones δ_A, δ_B, and δ_C, respectively. Terms of (25) associated to cones, other than $\delta_A, \delta_B, \delta_C$, in these decompositions, do not have a pole in s_0 and hence do not contribute to R_1.

Let us write down the seven contributions to the 'residue' R_1 explicitly. We obtain

$$\begin{aligned} R_1 = L_A(s_0) &\frac{\Sigma(\delta_A)(s_0)}{\left(p^{\sigma_2 + m_2 s_0} - 1\right)(p - 1)} + L_B(s_0) \frac{\Sigma(\delta_B)(s_0)}{\left(p^{\sigma_1 + m_1 s_0} - 1\right)(p - 1)} \\ &+ L_C(s_0) \frac{\Sigma(\delta_C)(s_0)}{\left(p^{\sigma_1 + m_1 s_0} - 1\right)\left(p^{\sigma_2 + m_2 s_0} - 1\right)} + L_{[AB]}(s_0) \frac{\Sigma(\Delta_{[AB]})(s_0)}{p - 1} \\ &+ L_{[AC]}(s_0) \frac{\Sigma(\Delta_{[AC]})(s_0)}{p^{\sigma_2 + m_2 s_0} - 1} + L_{[BC]}(s_0) \frac{\Sigma(\Delta_{[BC]})(s_0)}{p^{\sigma_1 + m_1 s_0} - 1} + L_{\tau_0}(s_0) \Sigma(\Delta_{\tau_0})(s_0). \end{aligned}$$

2.6. **The numbers N_τ.** Let us fix notations for the coefficients of f. We put

$$f(x,y,z) = \sum_{\omega=(\omega_1,\omega_2,\omega_3)\in\mathbf{Z}^3_{\geqslant 0}} a_\omega x^{\omega_1} y^{\omega_2} z^{\omega_3} \in \mathbf{Z}_p[x,y,z].$$

For $a \in \mathbf{Z}_p$, we denote by $\overline{a} = a + p\mathbf{Z}_p \in \mathbf{F}_p$ its reduction modulo $p\mathbf{Z}_p$. Recall that for every face τ of Γ_f, we have

$$f_\tau(x,y,z) = \sum_{\omega\in\mathbf{Z}^3\cap\tau} a_\omega x^{\omega_1} y^{\omega_2} z^{\omega_3} \quad \text{and} \quad \overline{f_\tau}(x,y,z) = \sum_{\omega\in\mathbf{Z}^3\cap\tau} \overline{a_\omega} x^{\omega_1} y^{\omega_2} z^{\omega_3}.$$

Because the polynomial f is non-degenerate over $\mathbf{F}_p$ with respect to all the compact faces of its Newton polyhedron (and thus especially with respect to the vertices A, B, C), we have that none of the numbers $\overline{a_A}, \overline{a_B}, \overline{a_C}$ equals zero.

Hence the numbers N_τ in the formula for Z_f^0 are as follows. For the vertices of τ_0 we find

$$N_A = \#\left\{(x,y,z) \in (\mathbf{F}_p^\times)^3 \;\middle|\; \overline{a_A} x^{x_A} y^{y_A} = 0\right\} = 0,$$

and analogously, $N_B = N_C = 0$. About the number $N_{[AB]}$ we don't know so much, except that

$$\begin{aligned} N_{[AB]} &= \#\left\{(x,y,z) \in (\mathbf{F}_p^\times)^3 \;\middle|\; \overline{f_{[AB]}}(x,y) = \overline{a_A} x^{x_A} y^{y_A} + \cdots + \overline{a_B} x^{x_B} y^{y_B} = 0\right\} \\ &= (p-1)N, \end{aligned}$$

with

$$N = \#\left\{(x,y) \in (\mathbf{F}_p^\times)^2 \;\middle|\; \overline{f_{[AB]}}(x,y) = 0\right\}.$$

For the other edges we find

$$N_{[AC]} = \#\left\{(x,y,z) \in (\mathbf{F}_p^\times)^3 \;\middle|\; \overline{a_A} x^{x_A} y^{y_A} + \overline{a_C} x^{x_C} y^{y_C} z = 0\right\} = (p-1)^2,$$

and analogously, $N_{[BC]} = (p-1)^2$. Finally, for τ_0 we obtain

$$N_{\tau_0} = \#\left\{(x,y,z) \in (\mathbf{F}_p^\times)^3 \;\middle|\; \overline{f_{[AB]}}(x,y) + \overline{a_C} x^{x_C} y^{y_C} z = 0\right\} = (p-1)^2 - N.$$

2.7. **The factors $L_\tau(s_0)$.** The above formulas for the N_τ give rise to the following expressions for the $L_\tau(s_0)$:

$$\begin{aligned} L_A(s_0) = L_B(s_0) = L_C(s_0) &= \left(\frac{p-1}{p}\right)^3, \\ L_{[AB]}(s_0) &= \left(\frac{p-1}{p}\right)^3 - \frac{(p-1)N}{p^2}\frac{p^{s_0}-1}{p^{s_0+1}-1}, \\ L_{[AC]}(s_0) = L_{[BC]}(s_0) &= \left(\frac{p-1}{p}\right)^3 - \left(\frac{p-1}{p}\right)^2\frac{p^{s_0}-1}{p^{s_0+1}-1}, \\ \text{and} \qquad L_{\tau_0}(s_0) &= \left(\frac{p-1}{p}\right)^3 - \frac{(p-1)^2-N}{p^2}\frac{p^{s_0}-1}{p^{s_0+1}-1}. \end{aligned}$$

2.8. **Multiplicities of the relevant simplicial cones.** We use Proposition 1.4 to calculate the multiplicities of the relevant simplicial cones (and their corresponding fundamental parallelepipeds), thereby exploiting the relations we obtained in Subsection 2.2. That way we find[22]

$$\mu_A = \operatorname{mult}\delta_A = \#H(v_0,v_2,v_3) = \left\|\begin{matrix} a_0 & b_0 & c_0\\ a_2 & b_2 & c_2\\ 0 & 0 & 1\end{matrix}\right\| = \begin{vmatrix} a_0 & b_0\\ a_2 & b_2\end{vmatrix} > 0,$$
$$\mu_B = \operatorname{mult}\delta_B = \#H(v_0,v_1,v_3) = \left\|\begin{matrix} a_0 & b_0 & c_0\\ a_1 & b_1 & c_1\\ 0 & 0 & 1\end{matrix}\right\| = -\begin{vmatrix} a_0 & b_0\\ a_1 & b_1\end{vmatrix} > 0,$$
$$\mu_C = \operatorname{mult}\delta_C = \#H(v_0,v_1,v_2) = \left\|\begin{matrix} a_0 & b_0 & c_0\\ a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\end{matrix}\right\| = \begin{vmatrix} a_0 & b_0 & c_0\\ a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\end{vmatrix} > 0$$

for the maximal dimensional simplicial cones, while for the two-dimensional cones we obtain

$$\begin{aligned}
\operatorname{mult}\Delta_{[AB]} = \#H(v_0,v_3) &= \gcd\left(\left\|\begin{matrix} a_0 & b_0\\ 0 & 0\end{matrix}\right\|, \left\|\begin{matrix} a_0 & c_0\\ 0 & 1\end{matrix}\right\|, \left\|\begin{matrix} b_0 & c_0\\ 0 & 1\end{matrix}\right\|\right)\\
&= \gcd(0,a_0,b_0) = 1,\\
\operatorname{mult}\Delta_{[AC]} = \#H(v_0,v_2) &= \gcd\left(\left\|\begin{matrix} a_0 & b_0\\ a_2 & b_2\end{matrix}\right\|, \left\|\begin{matrix} a_0 & c_0\\ a_2 & c_2\end{matrix}\right\|, \left\|\begin{matrix} b_0 & c_0\\ b_2 & b_2\end{matrix}\right\|\right)\\
&= \gcd\left(\begin{vmatrix} a_0 & b_0\\ a_2 & b_2\end{vmatrix}, |\beta_A|\begin{vmatrix} a_0 & b_0\\ a_2 & b_2\end{vmatrix}, |\alpha_A|\begin{vmatrix} a_0 & b_0\\ a_2 & b_2\end{vmatrix}\right)\\
&= \begin{vmatrix} a_0 & b_0\\ a_2 & b_2\end{vmatrix} = \mu_A, \qquad \text{and}\\
\operatorname{mult}\Delta_{[BC]} = \#H(v_0,v_1) &= \gcd\left(\left\|\begin{matrix} a_0 & b_0\\ a_1 & b_1\end{matrix}\right\|, \left\|\begin{matrix} a_0 & c_0\\ a_1 & c_1\end{matrix}\right\|, \left\|\begin{matrix} b_0 & c_0\\ b_1 & b_1\end{matrix}\right\|\right)\\
&= \gcd\left(-\begin{vmatrix} a_0 & b_0\\ a_1 & b_1\end{vmatrix}, -|\beta_B|\begin{vmatrix} a_0 & b_0\\ a_1 & b_1\end{vmatrix}, -|\alpha_B|\begin{vmatrix} a_0 & b_0\\ a_1 & b_1\end{vmatrix}\right)\\
&= -\begin{vmatrix} a_0 & b_0\\ a_1 & b_1\end{vmatrix} = \mu_B.
\end{aligned}$$

For the one-dimensional cone δ_{τ_0}, finally, we have of course that

$$\operatorname{mult}\Delta_{\tau_0} = \#H(v_0) = \gcd(a_0,b_0,c_0) = 1.$$

2.9. **The sums $\Sigma(\cdot)(s_0)$.** We found above that the multiplicities of $\Delta_{[AB]}$ and Δ_{τ_0} both equal one; i.e., their corresponding fundamental parallelepipeds contain only one integral point which must be the origin: $H(v_0,v_3) = H(v_0) = \{(0,0,0)\}$. Hence

$$\Sigma(\Delta_{[AB]})(s_0) = \Sigma(\Delta_{\tau_0})(s_0) = \sum_{h\in\{(0,0,0)\}} p^{\sigma(h)+m(h)s_0} = 1.$$

Furthermore we saw that the multiplicities of Δ_A and $\Delta_{[AC]}$ are equal:

$$\mu_A = \#H(v_0,v_2,v_3) = \#H(v_0,v_2).$$

[22] Cfr. Notation 1.5.

The inclusion $H(v_0, v_2, v_3) \supseteq H(v_0, v_2)$ thus implies equality:

$$H_A = H(v_0, v_2, v_3) = H(v_0, v_2),$$

and therefore,

$$\Sigma_A = \Sigma(\delta_A)(s_0) = \Sigma(\Delta_{[AC]})(s_0) = \sum_{h \in H_A} p^{\sigma(h)+m(h)s_0}.$$

Analogously we have

$$\begin{aligned} H_B = H(v_0, v_1, v_3) = H(v_0, v_1) \qquad \text{and} \\ \Sigma_B = \Sigma(\delta_B)(s_0) = \Sigma(\Delta_{[BC]})(s_0) = \sum_{h \in H_B} p^{\sigma(h)+m(h)s_0}. \end{aligned}$$

Consistently, we shall also denote

$$\begin{aligned} H_C = H(v_0, v_1, v_2) \qquad \text{and} \\ \Sigma_C = \Sigma(\delta_C)(s_0) = \sum_{h \in H_C} p^{\sigma(h)+m(h)s_0}. \end{aligned}$$

Note that, since $\overline{\Delta_{[AC]}}, \overline{\Delta_{[BC]}}, \overline{\delta_C} \subseteq \overline{\Delta_C}$, we have that[23]

$$m(h) = C \cdot h \qquad \text{for all} \qquad h \in H_A \cup H_B \cup H_C \subseteq \overline{\Delta_C}.$$

Hence, if we denote by w the vector

$$w = (1,1,1) + s_0(x_C, y_C, 1) \in \mathbf{C}^3,$$

it holds that

$$\Sigma_V = \sum_{h \in H_V} p^{w \cdot h}; \qquad V = A, B, C.$$

2.10. **A new formula for R_1.** If we denote

$$F_1 = p^{w \cdot v_1} - 1 = p^{\sigma_1 + m_1 s_0} - 1 \qquad \text{and} \qquad F_2 = p^{w \cdot v_2} - 1 = p^{\sigma_2 + m_2 s_0} - 1,$$

the results above on the numbers N_τ and the multiplicities of the cones lead to

$$\begin{aligned} R_1 = \left(\frac{p-1}{p}\right)^3 & \left[\frac{\Sigma_A}{F_2(p-1)} + \frac{\Sigma_B}{F_1(p-1)} + \frac{\Sigma_C}{F_1F_2} + \frac{1}{p-1} + \frac{\Sigma_A}{F_2} + \frac{\Sigma_B}{F_1} + 1\right] \\ & - \left(\frac{p-1}{p}\right)^2 \frac{p^{s_0}-1}{p^{s_0+1}-1} \left[\frac{N}{(p-1)^2} + \frac{\Sigma_A}{F_2} + \frac{\Sigma_B}{F_1} + \frac{(p-1)^2 - N}{(p-1)^2}\right]. \end{aligned}$$

If we put $R_1' = (p/(p-1))^3 R_1$, this formula can be simplified to

$$R_1' = \frac{1}{1 - p^{-s_0-1}} \left(\frac{\Sigma_A}{F_2} + \frac{\Sigma_B}{F_1} + 1\right) + \frac{\Sigma_C}{F_1F_2}. \tag{26}$$

Note that the number N disappears from the equation. In what follows we shall prove that $R_1' = 0$.

[23] In this text, by the dot product $w_1 \cdot w_2$ of two complex vectors $w_1(a_1, b_1, c_1), w_2(a_2, b_2, c_2) \in \mathbf{C}^3$, we mean $w_1 \cdot w_2 = a_1a_2 + b_1b_2 + c_1c_2$.

2.11. **Formulas for Σ_A and Σ_B.** As in Section 1, we will consider the set

$$H_C = H(v_0, v_1, v_2) = \mathbf{Z}^3 \cap \Diamond(v_0, v_1, v_2)$$

as an additive group, endowed with addition modulo the lattice

$$\Lambda(v_0, v_1, v_2) = \mathbf{Z}v_0 + \mathbf{Z}v_1 + \mathbf{Z}v_2.$$

In this way, $H_A = \mathbf{Z}^3 \cap \Diamond(v_0, v_2)$ and $H_B = \mathbf{Z}^3 \cap \Diamond(v_0, v_1)$ become subgroups of H_C that correspond to the subgroups H_1 and H_2 of H in Section 1.

From the description of the elements of these groups there, we know that there exist numbers $\xi_A \in \{0, \ldots, \mu_A - 1\}$ and $\xi_B \in \{0, \ldots, \mu_B - 1\}$ with $\gcd(\xi_A, \mu_A) = \gcd(\xi_B, \mu_B) = 1$, such that the μ_A points of H_A are precisely

$$\left\{\frac{i\xi_A}{\mu_A}\right\} v_0 + \frac{i}{\mu_A} v_2; \qquad i = 0, \ldots, \mu_A - 1; \tag{27}$$

while the μ_B points of H_B are given by

$$\left\{\frac{j\xi_B}{\mu_B}\right\} v_0 + \frac{j}{\mu_B} v_1; \qquad j = 0, \ldots, \mu_B - 1. \tag{28}$$

Recall that ξ_A and ξ_B are, as elements of $\{0, \ldots, \mu_A - 1\}$ and $\{0, \ldots, \mu_B - 1\}$, respectively, uniquely determined by

$$\xi_A v_0 + v_2 \in \mu_A \mathbf{Z}^3 \qquad \text{and} \qquad \xi_B v_0 + v_1 \in \mu_B \mathbf{Z}^3. \tag{29}$$

These descriptions allow us to find 'closed' formulas for Σ_A and Σ_B. We know that

$$\Sigma_A = \Sigma(\delta_A)(s_0) = \Sigma(\Delta_{[AC]})(s_0) = \sum_{h \in H_A} p^{\sigma(h) + m(h)s_0} = \sum_{h \in H_A} p^{w \cdot h},$$

with $w = (1,1,1) + s_0(x_C, y_C, 1)$. Note that since s_0 is a candidate pole associated to τ_0, we have that $p^{w \cdot v_0} = p^{\sigma_0 + m_0 s_0} = 1$. Hence $p^{a(w \cdot v_0)} = p^{\{a\}(w \cdot v_0)}$ for every real number a. So if we write h as $h = h_0 v_0 + h_2 v_2$, we obtain

$$\begin{aligned} \Sigma_A = \sum_{h \in H_A} p^{h_0(w \cdot v_0) + h_2(w \cdot v_2)} &= \sum_{i=0}^{\mu_A - 1} \left(p^{\frac{\xi_A(w \cdot v_0) + w \cdot v_2}{\mu_A}}\right)^i \\ &= \frac{p^{w \cdot v_2} - 1}{p^{\frac{\xi_A(w \cdot v_0) + w \cdot v_2}{\mu_A}} - 1} = \frac{F_2}{p^{\frac{\xi_A(w \cdot v_0) + w \cdot v_2}{\mu_A}} - 1}. \end{aligned} \tag{30}$$

Completely analogously we find

$$\Sigma_B = \frac{F_1}{p^{\frac{\xi_B(w \cdot v_0) + w \cdot v_1}{\mu_B}} - 1}. \tag{31}$$

2.12. **A formula for $\mu_C = \operatorname{mult} \delta_C$.** We know from Section 1 that $\mu_A \mu_B \mid \mu_C$. We will give a useful interpretation of the quotient $\mu_C / \mu_A \mu_B$. We have the following:

$$\begin{aligned} \mu_C &= \begin{vmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \\ &= -a_1 \begin{vmatrix} b_0 & c_0 \\ b_2 & c_2 \end{vmatrix} + b_1 \begin{vmatrix} a_0 & c_0 \\ a_2 & c_2 \end{vmatrix} - c_1 \begin{vmatrix} a_0 & b_0 \\ a_2 & b_2 \end{vmatrix}. \end{aligned}$$

Using the relations from Subsection 2.2, we continue:

$$\begin{aligned}\mu_C &= -a_1\alpha_A \begin{vmatrix} a_0 & b_0 \\ a_2 & b_2 \end{vmatrix} - b_1\beta_A \begin{vmatrix} a_0 & b_0 \\ a_2 & b_2 \end{vmatrix} - c_1 \begin{vmatrix} a_0 & b_0 \\ a_2 & b_2 \end{vmatrix} \\ &= -\mu_A(a_1\alpha_A + b_1\beta_A + c_1) \\ &= -\mu_A\left(v_1 \cdot \overrightarrow{AC}\right),\end{aligned}$$

and since $v_1 \perp \overrightarrow{BC}$, we obtain

$$\begin{aligned}\mu_C &= -\mu_A\left(v_1 \cdot \overrightarrow{AC} - v_1 \cdot \overrightarrow{BC}\right) \\ &= -\mu_A\left(v_1 \cdot \overrightarrow{AB}\right).\end{aligned}$$

Because the vector $\overrightarrow{AB}$ lies in the xy-plane and is perpendicular to v_0 and its coordinates have greatest common divisor φ_{AB} and we assume that $x_A > x_B$, it must hold that

$$\overrightarrow{AB} = \varphi_{AB}(-b_0, a_0, 0).$$

Hence

$$\mu_C = -\mu_A\varphi_{AB}(a_0b_1 - a_1b_0) = \mu_A\mu_B\varphi_{AB}.$$

Next, we will use this formula in describing the points of H_C.

2.13. **Description of the points of H_C.** It follows from (27) and (28) that the $\mu_A\mu_B$ points of the subgroup $H_A + H_B \cong H_A \oplus H_B$ of H_C are

$$\left\{\frac{i\xi_A\mu_B + j\xi_B\mu_A}{\mu_A\mu_B}\right\} v_0 + \frac{j}{\mu_B}v_1 + \frac{i}{\mu_A}v_2; \qquad i = 0, \ldots, \mu_A - 1; \quad j = 0, \ldots, \mu_B - 1.$$

We know that the v_2-coordinates h_2 of the points $h \in H_C$ belong to the set

$$\left\{0, \frac{1}{\mu_A\varphi_{AB}}, \frac{2}{\mu_A\varphi_{AB}}, \ldots, \frac{\mu_A\varphi_{AB} - 1}{\mu_A\varphi_{AB}}\right\},$$

and that every $l/\mu_A\varphi_{AB}$ in this set occurs μ_B times as the v_2-coordinate of a point in H_C, while every $h \in H_A + H_B$ has a v_2-coordinate of the form i/μ_A with $i \in \{0, \ldots, \mu_A - 1\}$, and every such i/μ_A is the v_2-coordinate of exactly μ_B points in $H_A + H_B$. (Analogously for the v_1-coordinates.)

In order to describe all the points of H_C in a way as we did in Section 1 for the points of H, we need to find a set of representatives for the elements of $H_C/(H_A + H_B)$. The φ_{AB} cosets of $H_A + H_B$ are characterised by constant $\{h_1\}_{1/\mu_B}$ and constant $\{h_2\}_{1/\mu_A}$, which can each take indeed φ_{AB} possible values.

From the discussion in Section 1, we know there exists a unique point $h^* \in H_C$ with v_2-coordinate $h_2^* = 1/\mu_A\varphi_{AB}$ and v_1-coordinate $h_1^* = \eta/\mu_B\varphi_{AB} < 1/\mu_B$, and that the φ_{AB} multiples $\{kh^*\}$; $k = 0, \ldots, \varphi_{AB} - 1$; of h^* in H_C make good representatives for the cosets of $H_A + H_B$. We will now try to find h^*.

If we denote by M the matrix

$$M = \begin{pmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}$$

with $\det M = \mu_C$, it follows from $(\operatorname{adj} M)M = (\det M)I = \mu_C I$ that

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} v_0 - \begin{vmatrix} a_0 & b_0 \\ a_2 & b_2 \end{vmatrix} v_1 + \begin{vmatrix} a_0 & b_0 \\ a_1 & b_1 \end{vmatrix} v_2 = \Psi v_0 - \mu_A v_1 - \mu_B v_2 = (0, 0, \mu_C),$$

and hence

$$h^* = \left\{\frac{-\Psi}{\mu_C}\right\} v_0 + \frac{1}{\mu_B\varphi_{AB}} v_1 + \frac{1}{\mu_A\varphi_{AB}} v_2 \in H_C$$

is the point we are looking for.

So, considering all possible sums (in the group H_C) of one of the $\mu_A\mu_B$ points

$$\left\{\frac{i\xi_A\mu_B + j\xi_B\mu_A}{\mu_A\mu_B}\right\} v_0 + \frac{j}{\mu_B} v_1 + \frac{i}{\mu_A} v_2; \qquad i = 0, \dots, \mu_A - 1; \quad j = 0, \dots, \mu_B - 1;$$

of $H_A + H_B$ and one of the φ_{AB} chosen representatives

$$\{kh^*\} = \left\{\frac{-k\Psi}{\mu_C}\right\} v_0 + \frac{k}{\mu_B\varphi_{AB}} v_1 + \frac{k}{\mu_A\varphi_{AB}} v_2; \qquad k = 0, \dots, \varphi_{AB} - 1;$$

for the cosets of $H_A + H_B$ in H_C, we find the $\mu_C = \mu_A\mu_B\varphi_{AB}$ points of H_C as

$$\left\{\frac{i\xi_A\mu_B\varphi_{AB} + j\xi_B\mu_A\varphi_{AB} - k\Psi}{\mu_C}\right\} v_0 + \frac{j\varphi_{AB} + k}{\mu_B\varphi_{AB}} v_1 + \frac{i\varphi_{AB} + k}{\mu_A\varphi_{AB}} v_2;$$
$$i = 0, \dots, \mu_A - 1; \quad j = 0, \dots, \mu_B - 1; \quad k = 0, \dots, \varphi_{AB} - 1.$$

Using the above description of the points of H_C, we will next derive a formula for Σ_C.

2.14. **A formula for Σ_C.** Recall that

$$\Sigma_C = \Sigma(\delta_C)(s_0) = \sum_{h \in H_C} p^{\sigma(h) + m(h)s_0} = \sum_{h \in H_C} p^{w \cdot h},$$

with $w = (1,1,1) + s_0(x_C, y_C, 1)$. If we write $h = h_0v_0 + h_1v_1 + h_2v_2$ and remember that $p^{w \cdot v_0} = 1$ and $\mu_C = \mu_A\mu_B\varphi_{AB}$, we find

$$\begin{aligned}
\Sigma_C &= \sum_{h \in H_C} p^{h_0(w \cdot v_0) + h_1(w \cdot v_1) + h_2(w \cdot v_2)} \\
&= \sum_{i=0}^{\mu_A - 1} \sum_{j=0}^{\mu_B - 1} \sum_{k=0}^{\varphi_{AB} - 1} p^{\frac{i\xi_A\mu_B\varphi_{AB} + j\xi_B\mu_A\varphi_{AB} - k\Psi}{\mu_C}(w \cdot v_0) + \frac{j\varphi_{AB} + k}{\mu_B\varphi_{AB}}(w \cdot v_1) + \frac{i\varphi_{AB} + k}{\mu_A\varphi_{AB}}(w \cdot v_2)} \\
&= \sum_i \Big(p^{\frac{\xi_A(w \cdot v_0) + w \cdot v_2}{\mu_A}}\Big)^i \sum_j \Big(p^{\frac{\xi_B(w \cdot v_0) + w \cdot v_1}{\mu_B}}\Big)^j \sum_k \Big(p^{\frac{-\Psi(w \cdot v_0) + \mu_A(w \cdot v_1) + \mu_B(w \cdot v_2)}{\mu_C}}\Big)^k \\
&= \frac{F_2}{p^{\frac{\xi_A(w \cdot v_0) + w \cdot v_2}{\mu_A}} - 1} \; \frac{F_1}{p^{\frac{\xi_B(w \cdot v_0) + w \cdot v_1}{\mu_B}} - 1} \; \frac{p^{\frac{-\Psi(w \cdot v_0) + \mu_A(w \cdot v_1) + \mu_B(w \cdot v_2)}{\mu_A\mu_B}} - 1}{p^{\frac{-\Psi(w \cdot v_0) + \mu_A(w \cdot v_1) + \mu_B(w \cdot v_2)}{\mu_C}} - 1}.
\end{aligned}$$

We already observed in Subsection 2.13 that if we put

$$M = \begin{pmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix},$$

the identity $(\operatorname{adj} M)M = (\det M)I = \mu_C I$ implies that

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} v_0 - \begin{vmatrix} a_0 & b_0 \\ a_2 & b_2 \end{vmatrix} v_1 + \begin{vmatrix} a_0 & b_0 \\ a_1 & b_1 \end{vmatrix} v_2 = \Psi v_0 - \mu_A v_1 - \mu_B v_2 = (0, 0, \mu_C). \tag{32}$$

Making the dot product with $w = (1,1,1) + s_0(x_C, y_C, 1)$ on all sides of the equation yields

$$-\Psi(w \cdot v_0) + \mu_A(w \cdot v_1) + \mu_B(w \cdot v_2) = \mu_C(-s_0 - 1).$$

Hence we find

$$\Sigma_C = \frac{F_1F_2}{p^{-s_0-1}-1}\frac{p^{\frac{-\Psi(w\cdot v_0)+\mu_A(w\cdot v_1)+\mu_B(w\cdot v_2)}{\mu_A\mu_B}}-1}{\left(p^{\frac{\xi_A(w\cdot v_0)+w\cdot v_2}{\mu_A}}-1\right)\left(p^{\frac{\xi_B(w\cdot v_0)+w\cdot v_1}{\mu_B}}-1\right)}. \tag{33}$$

2.15. **Proof of $R_1'=0$.** Bringing together Equations (26, 30, 31, 33) for R_1', Σ_A, Σ_B, and Σ_C, we obtain that

$$\begin{aligned}
R_1' &= \frac{1}{1-p^{-s_0-1}}\left(\frac{\Sigma_A}{F_2}+\frac{\Sigma_B}{F_1}+1\right)+\frac{\Sigma_C}{F_1F_2}\\
&= \frac{1}{1-p^{-s_0-1}}\left(\frac{1}{p^{\frac{\xi_A(w\cdot v_0)+w\cdot v_2}{\mu_A}}-1}+\frac{1}{p^{\frac{\xi_B(w\cdot v_0)+w\cdot v_1}{\mu_B}}-1}+1\right)\\
&\qquad\qquad + \frac{1}{p^{-s_0-1}-1}\frac{p^{\frac{-\Psi(w\cdot v_0)+\mu_A(w\cdot v_1)+\mu_B(w\cdot v_2)}{\mu_A\mu_B}}-1}{\left(p^{\frac{\xi_A(w\cdot v_0)+w\cdot v_2}{\mu_A}}-1\right)\left(p^{\frac{\xi_B(w\cdot v_0)+w\cdot v_1}{\mu_B}}-1\right)}\\
&= \frac{1}{p^{-s_0-1}-1}\frac{p^{\frac{(\xi_A\mu_B+\xi_B\mu_A)(w\cdot v_0)+\mu_A(w\cdot v_1)+\mu_B(w\cdot v_2)}{\mu_A\mu_B}}-1}{\left(p^{\frac{\xi_A(w\cdot v_0)+w\cdot v_2}{\mu_A}}-1\right)\left(p^{\frac{\xi_B(w\cdot v_0)+w\cdot v_1}{\mu_B}}-1\right)}\\
&\qquad\qquad - \frac{1}{1-p^{-s_0-1}}\frac{p^{\frac{-\Psi(w\cdot v_0)+\mu_A(w\cdot v_1)+\mu_B(w\cdot v_2)}{\mu_A\mu_B}}-1}{\left(p^{\frac{\xi_A(w\cdot v_0)+w\cdot v_2}{\mu_A}}-1\right)\left(p^{\frac{\xi_B(w\cdot v_0)+w\cdot v_1}{\mu_B}}-1\right)}.
\end{aligned}$$

Hence it is sufficient to prove that

$$p^{\frac{(\xi_A\mu_B+\xi_B\mu_A)(w\cdot v_0)}{\mu_A\mu_B}} = p^{-\frac{\Psi(w\cdot v_0)}{\mu_A\mu_B}},$$

or, as $p^{w\cdot v_0}=1$, equivalently, that

$$\frac{\xi_A\mu_B+\xi_B\mu_A+\Psi}{\mu_A\mu_B}\in\mathbf{Z}. \tag{34}$$

In fact, it follows from (29) and (32) that

$$\begin{aligned}
&(\xi_A\mu_B+\xi_B\mu_A+\Psi)v_0\\
&\qquad = \mu_B(\xi_Av_0+v_2)+\mu_A(\xi_Bv_0+v_1)+(\Psi v_0-\mu_Av_1-\mu_Bv_2)\in\mu_A\mu_B\mathbf{Z}^3.
\end{aligned}$$

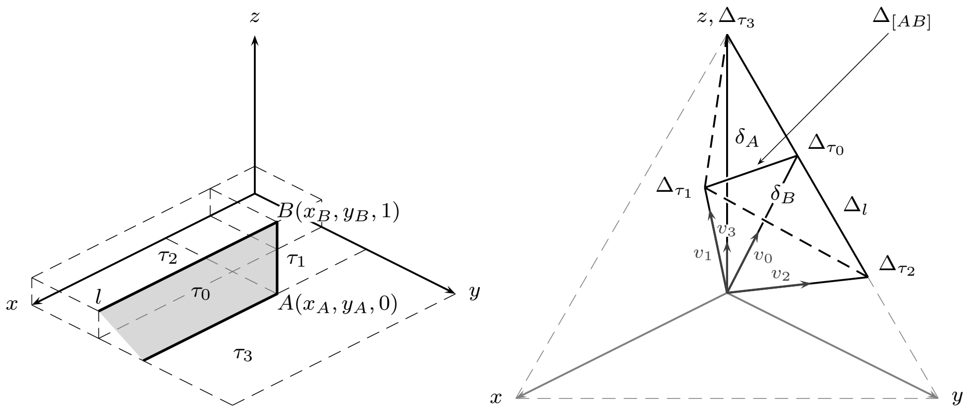
The primitivity of v_0 now implies (34), concluding Case I.

3. Case II: exactly one facet contributes to s_0 and this facet is a non-compact B_1-facet

3.1. **Figure and notations.** We shall assume that the one facet τ_0 contributing to s_0 is non-compact for the variable x, and B_1 with respect to the variable z. We denote by $A(x_A,y_A,0)$ the vertex of τ_0 in the xy-plane and by $B(x_B,y_B,1)$ the vertex in the plane $\{z=1\}$. The situation is sketched in Figure 5.

If we denote by $\overrightarrow{AB}(x_B-x_A,y_B-y_A,1)=(\alpha,\beta,1)$ the vector along the edge $[AB]$, then the unique primitive vector $v_0\in\mathbf{Z}^3_{\geqslant 0}$ perpendicular to τ_0 equals $v_0(0,1,-\beta)$, and an equation for the affine hull of τ_0 is given by

$$\operatorname{aff}(\tau_0)\leftrightarrow y-\beta z=y_A.$$



(a) Non-compact B_1-facet τ_0, its subfaces, and its neighbor facets τ_1, τ_2, and τ_3

(b) Relevant cones associated to relevant faces of Γ_f

FIGURE 5. Case II: the only facet contributing to s_0 is the non-compact B_1-facet τ_0

Note that since τ_0 is B_1, we must have $\beta < 0$ and hence $y_B < y_A$. The numerical data associated to τ_0 are therefore $(m(v_0), \sigma(v_0)) = (y_A, 1-\beta)$, and thus we assume

$$s_0 = \frac{\beta - 1}{y_A} + \frac{2n\pi i}{y_A \log p} \qquad \text{for some } n \in \mathbf{Z}.$$

We denote by τ_1 the facet of Γ_f that has the edge $[AB]$ in common with τ_0, by τ_2 the non-compact facet of Γ_f sharing with τ_0 a half-line with endpoint B, and finally, by τ_3 the facet lying in the xy-plane. Primitive vectors in $\mathbf{Z}^3_{\geqslant 0}$ perpendicular to τ_1, τ_2, τ_3 will be denoted by

$$v_1(a_1, b_1, c_1), \quad v_2(0, b_2, c_2), \quad v_3(0, 0, 1),$$

respectively, and equations for the affine supports of these facets are denoted

$$\begin{aligned} \operatorname{aff}(\tau_1) &\leftrightarrow a_1 x + b_1 y + c_1 z = m_1, \\ \operatorname{aff}(\tau_2) &\leftrightarrow \qquad\quad b_2 y + c_2 z = m_2, \\ \operatorname{aff}(\tau_3) &\leftrightarrow \qquad\qquad\qquad\; z = 0, \end{aligned}$$

for certain $m_1, m_2 \in \mathbf{Z}_{\geqslant 0}$. If we put $\sigma_1 = a_1 + b_1 + c_1$ and $\sigma_2 = b_2 + c_2$, then the numerical data for τ_1, τ_2, τ_3 are $(m_1, \sigma_1), (m_2, \sigma_2)$, and $(0, 1)$, respectively.

3.2. **The candidate pole s_0 and the contributions to its residue.** The aim of this section is to prove that s_0 is not a pole of Z^0_f. As in Case I (cfr. Subsection 2.4) it suffices to demonstrate that

$$R_1 = \lim_{s \to s_0} \left(p^{1-\beta+y_A s} - 1\right) Z^0_f(s) = 0.$$

Since we work with the local version of Igusa's p-adic zeta function, we only consider the compact faces of Γ_f in the formula for $Z^0_f(s)$ for non-degenerate f. Of course, in order to find an expression for R_1, we only need to account those compact faces that contribute to s_0, i.e., the compact subfaces A, B, and $[AB]$ of τ_0:

$$R_1 = \lim_{s \to s_0} \left(p^{1-\beta+y_A s} - 1\right) \sum_{\tau = A, B, [AB]} L_\tau(s) S(\Delta_\tau)(s).$$

As in Case I, we note that vertices A and B may be contained in facets other than τ_i; $i = 0, \ldots, 3$; and subsequently their associated cones Δ_A and Δ_B may be not simplicial. Therefore, instead of Δ_A and Δ_B, we shall consider the simplicial cones

$$\delta_A = \operatorname{cone}(v_0, v_1, v_3) \qquad \text{and} \qquad \delta_B = \operatorname{cone}(v_0, v_1, v_2)$$

as members of simplicial decompositions of Δ_A and Δ_B, respectively. It follows as before that of all cones in these decompositions, only δ_A and δ_B are relevant in the calculation of R_1:

$$\begin{aligned} R_1 = L_A(s_0) &\frac{\Sigma(\delta_A)(s_0)}{\left(p^{\sigma_1 + m_1 s_0} - 1\right)(p-1)} \\ &+ L_B(s_0) \frac{\Sigma(\delta_B)(s_0)}{\left(p^{\sigma_1 + m_1 s_0} - 1\right)\left(p^{\sigma_2 + m_2 s_0} - 1\right)} + L_{[AB]}(s_0) \frac{\Sigma(\Delta_{[AB]})(s_0)}{p^{\sigma_1 + m_1 s_0} - 1}. \end{aligned}$$

3.3. **The factors $L_\tau(s_0)$, the sums $\Sigma(\cdot)(s_0)$ and a new formula for R_1.** As in Case I we find easily that $N_A = N_B = 0$ and $N_{[AB]} = (p-1)^2$. Hence the factors $L_\tau(s_0)$ are as follows:

$$L_A(s_0) = L_B(s_0) = \left(\frac{p-1}{p}\right)^3 \qquad \text{and}$$
$$L_{[AB]}(s_0) = \left(\frac{p-1}{p}\right)^3 - \left(\frac{p-1}{p}\right)^2 \frac{p^{s_0} - 1}{p^{s_0+1} - 1}.$$

Let us look at the multiplicities of δ_A, δ_B, and $\Delta_{[AB]}$. For $\operatorname{mult} \delta_A$ we find

$$\mu_A = \operatorname{mult} \delta_A = \# H(v_0, v_1, v_3) = \left\| \begin{matrix} 0 & 1 & -\beta \\ a_1 & b_1 & c_1 \\ 0 & 0 & 1 \end{matrix} \right\| = a_1 > 0.$$

Although this non-compact edge does not appear in the formula for R_1, we also mention the multiplicity μ_l of the cone Δ_l associated to the half-line $l = \tau_0 \cap \tau_2$:

$$\mu_l = \operatorname{mult} \Delta_l = \# H(v_0(0, 1, -\beta), v_2(0, b_2, c_2)) = \left\| \begin{matrix} 1 & -\beta \\ b_2 & c_2 \end{matrix} \right\|.$$

Since the coordinate system (v_0, v_2) for the yz-plane has the opposite orientation of the coordinate system $(e_y(0,1,0), e_z(0,0,1))$ we work in, we have that

$$\mu_l = \left\| \begin{matrix} 1 & -\beta \\ b_2 & c_2 \end{matrix} \right\| = - \left| \begin{matrix} 1 & -\beta \\ b_2 & c_2 \end{matrix} \right| = -\beta b_2 - c_2 > 0.$$

We see now that

$$\begin{aligned} \mu_B = \operatorname{mult} \delta_B &= \# H(v_0, v_1, v_2) \\ &= \left\| \begin{matrix} 0 & 1 & -\beta \\ a_1 & b_1 & c_1 \\ 0 & b_2 & c_2 \end{matrix} \right\| = a_1 \left\| \begin{matrix} 1 & -\beta \\ b_2 & c_2 \end{matrix} \right\| = a_1(-\beta b_2 - c_2) = \mu_A \mu_l. \end{aligned}$$

Finally, for mult $\Delta_{[AB]}$ we obtain

$$\begin{aligned}\operatorname{mult}\Delta_{[AB]} = \#H(v_0,v_1) &= \gcd\left(\left\|\begin{matrix}0 & 1\\ a_1 & b_1\end{matrix}\right\|, \left\|\begin{matrix}0 & -\beta\\ a_1 & c_1\end{matrix}\right\|, \left\|\begin{matrix}1 & -\beta\\ b_1 & c_1\end{matrix}\right\|\right)\\ &= \gcd(a_1, -\beta a_1, |\beta b_1 + c_1|) = \gcd(a_1, -\beta a_1, |\alpha| a_1) = a_1 = \mu_A.\end{aligned}$$

In the third to last equality we used that $\beta b_1 + c_1 = -\alpha a_1$, which follows from the fact that $\overrightarrow{AB}(\alpha,\beta,1)$ is perpendicular to $v_1(a_1,b_1,c_1)$.

Since $H(v_0,v_1,v_3) \supseteq H(v_0,v_1)$ and $\mu_A = \#H(v_0,v_1,v_3) = \#H(v_0,v_1)$, we have that

$$H_A = H(v_0,v_1,v_3) = H(v_0,v_1),$$

and therefore,

$$\Sigma_A = \Sigma(\delta_A)(s_0) = \Sigma(\Delta_{[AB]})(s_0) = \sum_{h\in H_A} p^{\sigma(h)+m(h)s_0} = \sum_{h\in H_A} p^{w\cdot h},$$

with $w = (1,1,1) + s_0(x_B, y_B, 1) \in \mathbf{C}^3$. Furthermore we denote

$$H_l = H(v_0,v_2), \qquad H_B = H(v_0,v_1,v_2),$$

$$\Sigma_B = \Sigma(\delta_B)(s_0) = \sum_{h\in H_B} p^{\sigma(h)+m(h)s_0} = \sum_{h\in H_B} p^{w\cdot h},$$

$$F_1 = p^{w\cdot v_1} - 1 = p^{\sigma_1 + m_1 s_0} - 1, \qquad \text{and} \qquad F_2 = p^{w\cdot v_2} - 1 = p^{\sigma_2+m_2 s_0} - 1.$$

The considerations above result in the following concrete formula for R_1:

$$R_1 = \left(\frac{p-1}{p}\right)^3 \left[\frac{\Sigma_A}{F_1(p-1)} + \frac{\Sigma_B}{F_1F_2} + \frac{\Sigma_A}{F_1}\right] - \left(\frac{p-1}{p}\right)^2 \frac{p^{s_0}-1}{p^{s_0+1}-1}\frac{\Sigma_A}{F_1}.$$

With $R_1' = (p/(p-1))^3 R_1$, this can be simplified to

$$R_1' = \frac{1}{1-p^{-s_0-1}}\frac{\Sigma_A}{F_1} + \frac{\Sigma_B}{F_1F_2}. \tag{35}$$

Next, we will prove that $R_1' = 0$.

3.4. **Proof of $R_1' = 0$.** First, note that

$$-b_2 v_0 + v_2 = -b_2(0,1,-\beta) + (0,b_2,c_2) = -(0,0,-\beta b_2 - c_2) = -(0,0,\mu_l)$$

yields

$$\frac{-b_2}{\mu_l} v_0 + \frac{1}{\mu_l} v_2 = (0,0,-1) \in \mathbf{Z}^3 \quad \text{and} \quad p^{\frac{-b_2(w\cdot v_0)+w\cdot v_2}{\mu_l}} = p^{-s_0-1}, \tag{36}$$

with $w = (1,1,1) + s_0(x_B, y_B, 1)$.

Let us, as before, consider

$$H_B = H(v_0,v_1,v_2) = \mathbf{Z}^3 \cap \Diamond(v_0,v_1,v_2)$$

as a group, endowed with addition modulo $\mathbf{Z}v_0 + \mathbf{Z}v_1 + \mathbf{Z}v_2$. Then, by (36) and Theorem 1.7, there exists a $\xi_A \in \{0,\ldots,\mu_A - 1\}$ such that the elements of the subgroups $H_A = H(v_0,v_1)$ and $H_l = H(v_0,v_2)$ of H_B are given by

$$\left\{\frac{i\xi_A}{\mu_A}\right\} v_0 + \frac{i}{\mu_A} v_1; \qquad i = 0,\ldots,\mu_A - 1;$$

and

$$\left\{\frac{-jb_2}{\mu_l}\right\} v_0 + \frac{j}{\mu_l} v_2; \qquad j = 0,\ldots,\mu_l - 1;$$

respectively.

Furthermore, we found above that in this special case

$$\#H_B = \mu_B = \mu_A\mu_l = \#H_A\#H_l.$$

Hence $H_A \cap H_l = \{(0,0,0)\}$ implies that $H_B = H_A + H_l \cong H_A \oplus H_l$ and its elements are the following:

$$\left\{\frac{i\xi_A\mu_l - jb_2\mu_A}{\mu_A\mu_l}\right\}v_0 + \frac{i}{\mu_A}v_1 + \frac{j}{\mu_l}v_2; \qquad i = 0,\ldots,\mu_A - 1; \quad j = 0,\ldots,\mu_l - 1.$$

We can now easily calculate Σ_A and Σ_B. If, for $h \in H_B$, we denote by (h_0, h_1, h_2) the coordinates of h with respect to the basis (v_0, v_1, v_2) and keep in mind that $p^{w\cdot v_0} = 1$, we obtain

$$\begin{aligned}\Sigma_A &= \sum_{h\in H_A} p^{w\cdot h} = \sum_h p^{h_0(w\cdot v_0)+h_1(w\cdot v_1)} \\ &= \sum_{i=0}^{\mu_A-1}\left(p^{\frac{\xi_A(w\cdot v_0)+w\cdot v_1}{\mu_A}}\right)^i = \frac{p^{w\cdot v_1}-1}{p^{\frac{\xi_A(w\cdot v_0)+w\cdot v_1}{\mu_A}}-1} = \frac{F_1}{p^{\frac{\xi_A(w\cdot v_0)+w\cdot v_1}{\mu_A}}-1},\end{aligned} \tag{37}$$

while Σ_B is given by

$$\begin{aligned}\Sigma_B &= \sum_{h\in H_B} p^{w\cdot h} \\ &= \sum_h p^{h_0(w\cdot v_0)+h_1(w\cdot v_1)+h_2(w\cdot v_2)} \\ &= \sum_{i=0}^{\mu_A-1}\sum_{j=0}^{\mu_l-1} p^{\frac{i\xi_A\mu_l - jb_2\mu_A}{\mu_A\mu_l}(w\cdot v_0)+\frac{i}{\mu_A}(w\cdot v_1)+\frac{j}{\mu_l}(w\cdot v_2)} \\ &= \sum_i\left(p^{\frac{\xi_A(w\cdot v_0)+w\cdot v_1}{\mu_A}}\right)^i \sum_j\left(p^{\frac{-b_2(w\cdot v_0)+w\cdot v_2}{\mu_l}}\right)^j \\ &= \frac{F_1}{p^{\frac{\xi_A(w\cdot v_0)+w\cdot v_1}{\mu_A}}-1}\,\frac{F_2}{p^{\frac{-b_2(w\cdot v_0)+w\cdot v_2}{\mu_l}}-1} \\ &= \frac{1}{p^{-s_0-1}-1}\,\frac{F_1F_2}{p^{\frac{\xi_A(w\cdot v_0)+w\cdot v_1}{\mu_A}}-1},\end{aligned} \tag{38}$$

where we used (36) in the last step.

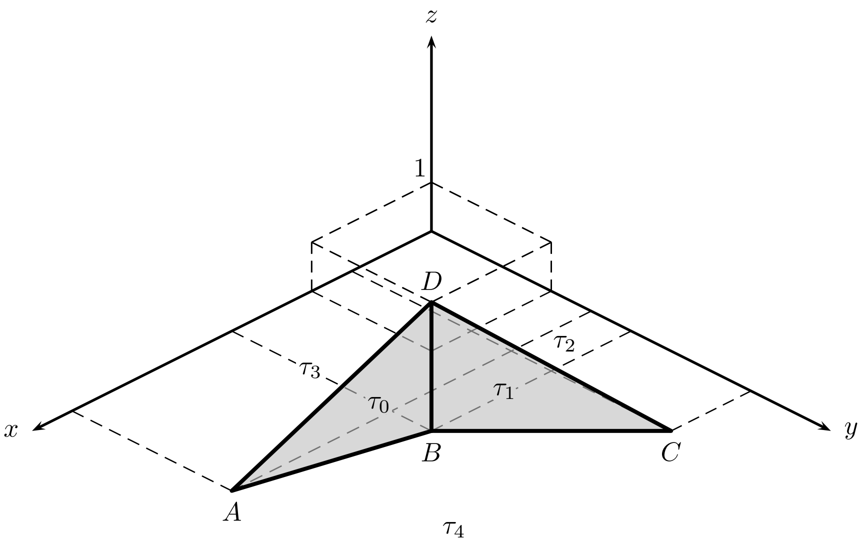
By Equations (35, 37, 38) we have $R_1' = 0$. This concludes Case II.

4. Case III: exactly two facets of Γ_f contribute to s_0, and these two facets are both B_1-simplices with respect to a same variable and have an edge in common

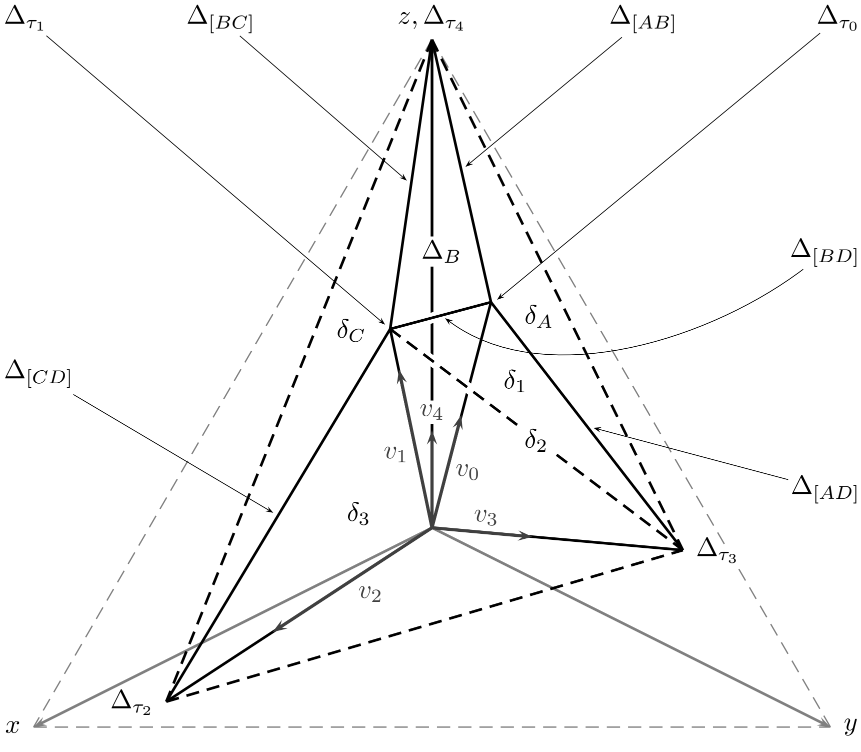
4.1. **Figure and notations.** Without loss of generality, we may assume that the B_1-simplices τ_0 and τ_1 contributing to s_0 are as drawn in Figure 6.

Let us fix notations. We denote, as indicated in Figure 6, the vertices of τ_0 and τ_1 and their coordinates by

$$A(x_A, y_A, 0), \quad B(x_B, y_B, 0), \quad C(x_C, y_C, 0), \quad \text{and} \quad D(x_D, y_D, 1).$$



(a) B_1-simplices τ_0 and τ_1, their subfaces and neighbor facets τ_2, τ_3, and τ_4



(b) Relevant cones associated to relevant faces of Γ_f

FIGURE 6. Case III: the only facets contributing to s_0 are the B_1-simplices τ_0 and τ_1

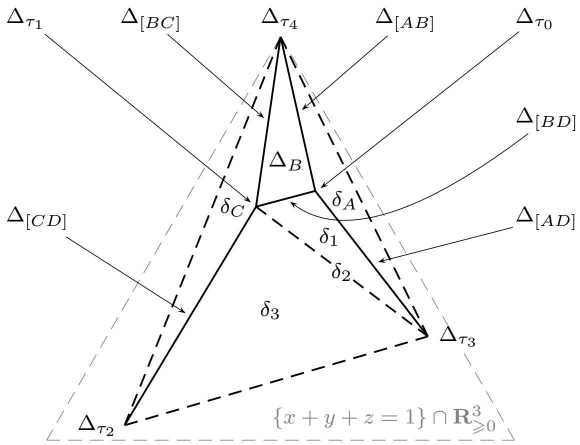


FIGURE 7. Sketch of the intersection of the contributing cones with the plane $\{x+y+z=1\}$

We denote the neighbor facets of τ_0 and τ_1 by τ_2, τ_3, τ_4. The unique primitive vectors perpendicular to τ_i; $i = 0, \dots, 4$; will be denoted by

$$v_0(a_0, b_0, c_0), \quad v_1(a_1, b_1, c_1), \quad v_2(a_2, b_2, c_2), \quad v_3(a_3, b_3, c_3), \quad v_4(0, 0, 1),$$

respectively. In this way the affine supports of these facets have equations of the form

$$\begin{aligned} \operatorname{aff}(\tau_i) &\leftrightarrow a_i x + b_i y + c_i z = m_i; \qquad i = 0, \dots, 3; \\ \operatorname{aff}(\tau_4) &\leftrightarrow \qquad\qquad\qquad\quad z = 0; \end{aligned}$$

and we associate to them the numerical data

$$\begin{aligned} (m_i, \sigma_i) &= (m(v_i), \sigma(v_i)) \ = (a_i x_D + b_i y_D + c_i, a_i + b_i + c_i); \qquad i = 0, \dots, 3; \\ (m_4, \sigma_4) &= (m(v_4), \sigma(v_4)) = (0, 1). \end{aligned}$$

We assume that τ_0 and τ_1 both contribute to the candidate pole s_0. With the present notations, this is, we assume that $p^{\sigma_0 + m_0 s_0} = p^{\sigma_1 + m_1 s_0} = 1$, or equivalently,

$$\begin{aligned} \Re(s_0) &= -\frac{\sigma_0}{m_0} = -\frac{\sigma_1}{m_1} = -\frac{a_0 + b_0 + c_0}{a_0 x_D + b_0 y_D + c_0} = -\frac{a_1 + b_1 + c_1}{a_1 x_D + b_1 y_D + c_1} \qquad \text{and} \\ \Im(s_0) &= \frac{2n\pi}{\gcd(m_0, m_1) \log p} \qquad \text{for some } n \in \mathbf{Z}. \end{aligned}$$

Throughout this section we will consider the following thirteen simplicial cones:

$$\begin{aligned} \delta_A &= \operatorname{cone}(v_0, v_3, v_4), & \Delta_{[AB]} &= \operatorname{cone}(v_0, v_4), & \Delta_{\tau_0} &= \operatorname{cone}(v_0), \\ \Delta_B &= \operatorname{cone}(v_0, v_1, v_4), & \Delta_{[BC]} &= \operatorname{cone}(v_1, v_4), & \Delta_{\tau_1} &= \operatorname{cone}(v_1). \\ \delta_C &= \operatorname{cone}(v_1, v_2, v_4), & \Delta_{[AD]} &= \operatorname{cone}(v_0, v_3), \\ \delta_1 &= \operatorname{cone}(v_0, v_1, v_3), & \Delta_{[BD]} &= \operatorname{cone}(v_0, v_1), \\ \delta_2 &= \operatorname{cone}(v_1, v_3), & \Delta_{[CD]} &= \operatorname{cone}(v_1, v_2), \\ \delta_3 &= \operatorname{cone}(v_1, v_2, v_3), \end{aligned}$$

The Δ_τ listed above are the simplicial cones associated to the faces τ. The cones $\Delta_A, \Delta_C, \Delta_D$, associated to the respective vertices A, C, D, are generally not simplicial. Later in this section we will consider simplicial subdivisions (without creating new rays) of Δ_A, Δ_C, and Δ_D that include $\{\delta_A\}, \{\delta_C\}$, and $\{\delta_1, \delta_2, \delta_3\}$, respectively (cfr. Figure 7).

Finally, let us fix notations for the vectors along the edges of τ_0 and τ_1:

$$\begin{aligned}
&\overrightarrow{AD}(x_D - x_A, y_D - y_A, 1) = (\alpha_A, \beta_A, 1),\\
&\overrightarrow{BD}(x_D - x_B, y_D - y_B, 1) = (\alpha_B, \beta_B, 1),\\
&\overrightarrow{CD}(x_D - x_C, y_D - y_C, 1) = (\alpha_C, \beta_C, 1),\\
&\overrightarrow{AB}(x_B - x_A, y_B - y_A, 0) = (\alpha_A - \alpha_B, \beta_A - \beta_B, 0),\\
&\overrightarrow{BC}(x_C - x_B, y_C - y_B, 0) = (\alpha_B - \alpha_C, \beta_B - \beta_C, 0).
\end{aligned}$$

The first three vectors are primitive; the last two are generally not. We put

$$\varphi_{AB} = \gcd(x_B - x_A, y_B - y_A) \qquad \text{and} \qquad \varphi_{BC} = \gcd(x_C - x_B, y_C - y_B).$$

4.2. **Some relations between the variables.** In the same way as in Case I we obtain that

$$\begin{aligned}
\begin{pmatrix} c_0 \\ c_3 \end{pmatrix} &= -\alpha_A \begin{pmatrix} a_0 \\ a_3 \end{pmatrix} - \beta_A \begin{pmatrix} b_0 \\ b_3 \end{pmatrix},\\
\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} &= -\alpha_B \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} - \beta_B \begin{pmatrix} b_0 \\ b_1 \end{pmatrix},\\
\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} &= -\alpha_C \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} - \beta_C \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.
\end{aligned}$$

A first consequence is that

$$\gcd(a_i, b_i, c_i) = \gcd(a_i, b_i) = 1; \qquad i = 0, \ldots, 3.$$

As a second consequence, we have

$$\begin{aligned}
\begin{vmatrix} a_0 & c_0 \\ a_3 & c_3 \end{vmatrix} &= -\beta_A \begin{vmatrix} a_0 & b_0 \\ a_3 & b_3 \end{vmatrix}, & \qquad \begin{vmatrix} b_0 & c_0 \\ b_3 & c_3 \end{vmatrix} &= \alpha_A \begin{vmatrix} a_0 & b_0 \\ a_3 & b_3 \end{vmatrix},\\
\begin{vmatrix} a_0 & c_0 \\ a_1 & c_1 \end{vmatrix} &= -\beta_B \begin{vmatrix} a_0 & b_0 \\ a_1 & b_1 \end{vmatrix}, & \qquad \begin{vmatrix} b_0 & c_0 \\ b_1 & c_1 \end{vmatrix} &= \alpha_B \begin{vmatrix} a_0 & b_0 \\ a_1 & b_1 \end{vmatrix},\\
\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} &= -\beta_C \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, & \qquad \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} &= \alpha_C \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.
\end{aligned}$$

In the calculations that will follow it is often convenient (or necessary) to know the signs of certain determinants. Coordinate system orientation considerations show that

$$\begin{array}{ccc}
\begin{vmatrix} a_0 & b_0 \\ a_3 & b_3 \end{vmatrix} > 0, & \begin{vmatrix} a_0 & b_0 \\ a_1 & b_1 \end{vmatrix} < 0, & \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} < 0,\\
\Psi = \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} > 0, & -\Omega = \begin{vmatrix} a_0 & b_0 \\ a_2 & b_2 \end{vmatrix} < 0, & \Theta = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} > 0,\\
\begin{vmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} > 0, & \begin{vmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} > 0, & \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} > 0.
\end{array} \tag{39}$$

4.3. **Igusa's local zeta function.** For the convenience of the reader, we recall the formula for the local Igusa zeta function Z_f^0 of f from Subsection 2.3:

$$Z_f^0 = \sum_{\substack{\tau \text{ compact} \\ \text{face of } \Gamma_f}} L_\tau S(\Delta_\tau), \tag{40}$$

with

$$L_\tau : s \mapsto L_\tau(s) = \left(\frac{p-1}{p}\right)^3 - \frac{N_\tau}{p^2} \frac{p^s - 1}{p^{s+1} - 1},$$
$$N_\tau = \#\left\{(x, y, z) \in (\mathbf{F}_p^\times)^3 \mid \overline{f_\tau}(x, y, z) = 0\right\},$$

and

$$\begin{aligned} S(\Delta_\tau) : s \mapsto S(\Delta_\tau)(s) &= \sum_{k \in \mathbf{Z}^3 \cap \Delta_\tau} p^{-\sigma(k) - m(k)s} \\ &= \sum_{i \in I} \frac{\Sigma(\delta_i)(s)}{\prod_{j \in J_i} (p^{\sigma(w_j) + m(w_j)s} - 1)}. \end{aligned} \tag{41}$$

Here $\{\delta_i\}_{i \in I}$ denotes a simplicial decomposition without introducing new rays of the cone Δ_τ associated to τ. The simplicial cone δ_i is supposed to be strictly positively spanned by the linearly independent primitive vectors w_j, $j \in J_i$, in $\mathbf{Z}^n_{\geqslant 0} \setminus \{0\}$, and $\Sigma(\delta_i)$ is the function

$$\Sigma(\delta_i) : s \mapsto \Sigma(\delta_i)(s) = \sum_h p^{\sigma(h) + m(h)s},$$

where h runs through the elements of the set

$$H(w_j)_{j \in J_i} = \mathbf{Z}^3 \cap \lozenge(w_j)_{j \in J_i},$$

with

$$\lozenge(w_j)_{j \in J_i} = \left\{\sum\nolimits_{j \in J_i} h_j w_j \;\middle|\; h_j \in [0, 1) \text{ for all } j \in J_i\right\}$$

the fundamental parallelepiped spanned by the vectors w_j, $j \in J_i$.

4.4. **The candidate pole s_0 and its residues.**

Remark 4.1. This subsection is totally general to all cases where s_0 is a candidate pole of expected order two, i.e., to Cases III–V.

We want to prove that s_0 is not a pole of Z_f^0. Since s_0 is a candidate pole of expected order two (and therefore is a pole of actual order at most two), it is enough to prove that the coefficients a_{-2} and a_{-1} in the Laurent series

$$Z_f^0(s) = \sum_{k=-2}^{\infty} a_k (s - s_0)^k$$

of Z_f^0 centered at s_0, both equal zero. These coefficients are given by

$$a_{-2} = \lim_{s \to s_0} (s - s_0)^2 Z_f^0(s) \qquad \text{and}$$
$$a_{-1} = \operatorname{Res}(Z_f^0, s_0) = \lim_{s \to s_0} \frac{d}{ds} \left[(s - s_0)^2 Z_f^0(s)\right].$$

Alternatively (and consequently), it is sufficient to show that

$$\begin{aligned} R_2 &= \lim_{s\to s_0} \left(p^{\sigma_0+m_0 s}-1\right)\left(p^{\sigma_1+m_1 s}-1\right) Z_f^0(s) \\ &= (\log p)^2 m_0 m_1 a_{-2} \end{aligned}$$

and

$$\begin{aligned} R_1 &= \lim_{s\to s_0} \frac{d}{ds}\left[\left(p^{\sigma_0+m_0 s}-1\right)\left(p^{\sigma_1+m_1 s}-1\right) Z_f^0(s)\right] \\ &= (\log p)^2 m_0 m_1 a_{-1} + \frac{1}{2}(\log p)^3 m_0 m_1 (m_0+m_1) a_{-2} \end{aligned}$$

both vanish. We will in the rest of this section prove that $R_2 = R_1 = 0$.

4.5. **Terms contributing to R_2 and R_1.** We intend to calculate R_2 and R_1 based on Formula (40) for Z_f^0.

Precisely 11 compact faces of Γ_f contribute to the candidate pole s_0. These are the subfaces $A, B, C, D, [AB], [BC], [AD], [BD], [CD], \tau_0$, and τ_1 of the two compact facets τ_0 and τ_1 that have s_0 as an associated candidate pole. They are only the terms of (40) associated to these faces that should be taken into account in the calculation of R_1. The other terms do not have s_0 as a pole and therefore do not contribute to the limit R_1.

Vertex B is only contained in the facets τ_0, τ_1, and τ_4; hence its associated cone Δ_B is simplicial. The cones associated to the other vertices A, C, and D are generally not simplicial. For dealing with S_A and S_C, we will, just as in Case I, consider simplicial decompositions of Δ_A and Δ_C that contain δ_A and δ_C, respectively. Terms of (41) associated to other cones than δ_A and δ_C in these decompositions do not have a pole in s_0, hence do not contribute to R_1. Vertex D is contained in at least four facets. We shall consider a decomposition of Δ_D into simplicial cones among which δ_1, δ_2, and δ_3. Only the terms associated to these three cones should be taken into account when calculating R_1. This makes a total of 13 terms contributing to R_1 (three coming from D and one for every other contributing face).

The limit R_2 counts fewer contributions: the only terms of (40) and (41) that need to be considered are the ones that have a double pole in s_0. These are the terms associated to $B, [BD]$, and δ_1. All other terms have at most a single pole in s_0 and do not contribute to R_2.

Let us write down these contributions explicitly. For R_2 we obtain

$$R_2 = L_B(s_0)\frac{\Sigma(\Delta_B)(s_0)}{p-1} + L_D(s_0)\frac{\Sigma(\delta_1)(s_0)}{p^{\sigma_3+m_3 s_0}-1} + L_{[BD]}(s_0)\Sigma(\Delta_{[BD]})(s_0).$$

The thirteen terms making up R_1 are

$$\begin{aligned} R_1 = \frac{d}{ds}&\left[L_A(s)\frac{\left(p^{\sigma_1+m_1 s}-1\right)\Sigma(\delta_A)(s)}{\left(p^{\sigma_3+m_3 s}-1\right)(p-1)}\right]\Bigg|_{s=s_0} + \frac{d}{ds}\left[L_B(s)\frac{\Sigma(\Delta_B)(s)}{p-1}\right]\Bigg|_{s=s_0} \\ &+ \frac{d}{ds}\left[L_C(s)\frac{\left(p^{\sigma_0+m_0 s}-1\right)\Sigma(\delta_C)(s)}{\left(p^{\sigma_2+m_2 s}-1\right)(p-1)}\right]\Bigg|_{s=s_0} + \frac{d}{ds}\left[L_D(s)\frac{\Sigma(\delta_1)(s)}{p^{\sigma_3+m_3 s}-1}\right]\Bigg|_{s=s_0} \\ &+ \frac{d}{ds}\left[L_D(s)\frac{\left(p^{\sigma_0+m_0 s}-1\right)\Sigma(\delta_2)(s)}{p^{\sigma_3+m_3 s}-1}\right]\Bigg|_{s=s_0} \end{aligned}$$

$$
\begin{aligned}
&+\frac{d}{ds}\left[L_D(s)\frac{\big(p^{\sigma_0+m_0s}-1\big)\Sigma(\delta_3)(s)}{\big(p^{\sigma_2+m_2s}-1\big)\big(p^{\sigma_3+m_3s}-1\big)}\right]\Bigg|_{s=s_0}\\
&+\frac{d}{ds}\left[L_{[AB]}(s)\frac{\big(p^{\sigma_1+m_1s}-1\big)\Sigma(\Delta_{[AB]})(s)}{p-1}\right]\Bigg|_{s=s_0}\\
&+\frac{d}{ds}\left[L_{[BC]}(s)\frac{\big(p^{\sigma_0+m_0s}-1\big)\Sigma(\Delta_{[BC]})(s)}{p-1}\right]\Bigg|_{s=s_0}\\
&+\frac{d}{ds}\left[L_{[AD]}(s)\frac{\big(p^{\sigma_1+m_1s}-1\big)\Sigma(\Delta_{[AD]})(s)}{p^{\sigma_3+m_3s}-1}\right]\Bigg|_{s=s_0}\\
&\qquad+\frac{d}{ds}\left[L_{[BD]}(s)\Sigma(\Delta_{[BD]})(s)\right]\big|_{s=s_0}\\
&+\frac{d}{ds}\left[L_{[CD]}(s)\frac{\big(p^{\sigma_0+m_0s}-1\big)\Sigma(\Delta_{[CD]})(s)}{p^{\sigma_2+m_2s}-1}\right]\Bigg|_{s=s_0}\\
&\quad+\frac{d}{ds}\left[L_{\tau_0}(s)\big(p^{\sigma_1+m_1s}-1\big)\Sigma(\Delta_{\tau_0})(s)\right]\big|_{s=s_0}\\
&\qquad\qquad+\frac{d}{ds}\left[L_{\tau_1}(s)\big(p^{\sigma_0+m_0s}-1\big)\Sigma(\Delta_{\tau_1})(s)\right]\big|_{s=s_0}.
\end{aligned}
$$

After simplification, R_1 is given by

$$
\begin{aligned}
R_1 &= L_A(s_0)\frac{m_1(\log p)\Sigma(\delta_A)(s_0)}{\big(p^{\sigma_3+m_3s_0}-1\big)(p-1)}+L_B'(s_0)\frac{\Sigma(\Delta_B)(s_0)}{p-1}+L_B(s_0)\frac{\Sigma(\Delta_B)'(s_0)}{p-1}\\
&+L_C(s_0)\frac{m_0(\log p)\Sigma(\delta_C)(s_0)}{\big(p^{\sigma_2+m_2s_0}-1\big)(p-1)}+L_D'(s_0)\frac{\Sigma(\delta_1)(s_0)}{p^{\sigma_3+m_3s_0}-1}+L_D(s_0)\frac{\Sigma(\delta_1)'(s_0)}{p^{\sigma_3+m_3s_0}-1}\\
&-L_D(s_0)\frac{m_3(\log p)p^{\sigma_3+m_3s_0}\Sigma(\delta_1)(s_0)}{\big(p^{\sigma_3+m_3s_0}-1\big)^2}+L_D(s_0)\frac{m_0(\log p)\Sigma(\delta_2)(s_0)}{p^{\sigma_3+m_3s_0}-1}\\
&+L_D(s_0)\frac{m_0(\log p)\Sigma(\delta_3)(s_0)}{\big(p^{\sigma_2+m_2s_0}-1\big)\big(p^{\sigma_3+m_3s_0}-1\big)}\\
&+L_{[AB]}(s_0)\frac{m_1(\log p)\Sigma(\Delta_{[AB]})(s_0)}{p-1}+L_{[BC]}(s_0)\frac{m_0(\log p)\Sigma(\Delta_{[BC]})(s_0)}{p-1}\\
&+L_{[AD]}(s_0)\frac{m_1(\log p)\Sigma(\Delta_{[AD]})(s_0)}{p^{\sigma_3+m_3s_0}-1}+L_{[BD]}'(s_0)\Sigma(\Delta_{[BD]})(s_0)\\
&+L_{[BD]}(s_0)\Sigma(\Delta_{[BD]})'(s_0)+L_{[CD]}(s_0)\frac{m_0(\log p)\Sigma(\Delta_{[CD]})(s_0)}{p^{\sigma_2+m_2s_0}-1}\\
&+L_{\tau_0}(s_0)m_1(\log p)\Sigma(\Delta_{\tau_0})(s_0)+L_{\tau_1}(s_0)m_0(\log p)\Sigma(\Delta_{\tau_1})(s_0).
\end{aligned}
$$

4.6. **The numbers N_τ.** Analogously to Case I we obtain

$$
\begin{gathered}
N_A=N_B=N_C=N_D=0,\\
N_{[AB]}=(p-1)N_0,\qquad N_{[BC]}=(p-1)N_1,\\
N_{[AD]}=N_{[BD]}=N_{[CD]}=(p-1)^2,\\
N_{\tau_0}=(p-1)^2-N_0,\qquad N_{\tau_1}=(p-1)^2-N_1,
\end{gathered}
$$

with

$$N_0 = \#\left\{(x,y) \in (\mathbf{F}_p^\times)^2 \mid \overline{f_{[AB]}}(x,y) = 0\right\},$$
$$N_1 = \#\left\{(x,y) \in (\mathbf{F}_p^\times)^2 \mid \overline{f_{[BC]}}(x,y) = 0\right\}.$$

4.7. **The factors $L_\tau(s_0)$ and $L'_\tau(s_0)$.** For the $L_\tau(s_0)$ we obtain

$$L_A(s_0) = L_B(s_0) = L_C(s_0) = L_D(s_0) = \left(\frac{p-1}{p}\right)^3,$$
$$L_{[AB]}(s_0) = \left(\frac{p-1}{p}\right)^3 - \frac{(p-1)N_0}{p^2}\frac{p^{s_0}-1}{p^{s_0+1}-1},$$
$$L_{[BC]}(s_0) = \left(\frac{p-1}{p}\right)^3 - \frac{(p-1)N_1}{p^2}\frac{p^{s_0}-1}{p^{s_0+1}-1},$$
$$L_{[AD]}(s_0) = L_{[BD]}(s_0) = L_{[CD]}(s_0) = \left(\frac{p-1}{p}\right)^3 - \left(\frac{p-1}{p}\right)^2\frac{p^{s_0}-1}{p^{s_0+1}-1},$$
$$L_{\tau_0}(s_0) = \left(\frac{p-1}{p}\right)^3 - \frac{(p-1)^2-N_0}{p^2}\frac{p^{s_0}-1}{p^{s_0+1}-1},$$
$$L_{\tau_1}(s_0) = \left(\frac{p-1}{p}\right)^3 - \frac{(p-1)^2-N_1}{p^2}\frac{p^{s_0}-1}{p^{s_0+1}-1},$$

while the $L'_\tau(s_0)$ are given by

$$L'_B(s_0) = L'_D(s_0) = 0,$$
$$L'_{[BD]}(s_0) = -(\log p)\left(\frac{p-1}{p}\right)^3\frac{p^{s_0+1}}{\left(p^{s_0+1}-1\right)^2}.$$

4.8. **Multiplicities of the relevant simplicial cones.** Based on Proposition 1.4 and the relations obtained in Subsection 4.2, we have, analogously to Case I, that

$$\operatorname{mult}\Delta_{[AB]} = \operatorname{mult}\Delta_{[BC]} = \operatorname{mult}\Delta_{\tau_0} = \operatorname{mult}\Delta_{\tau_1} = 1,$$
$$\mu_A = \operatorname{mult}\delta_A = \#H(v_0,v_3,v_4) = \operatorname{mult}\Delta_{[AD]} = \#H(v_0,v_3) = \begin{vmatrix} a_0 & b_0 \\ a_3 & b_3 \end{vmatrix} > 0,$$
$$\mu_B = \operatorname{mult}\Delta_B = \#H(v_0,v_1,v_4) = \operatorname{mult}\Delta_{[BD]} = \#H(v_0,v_1) = -\begin{vmatrix} a_0 & b_0 \\ a_1 & b_1 \end{vmatrix} > 0,$$
$$\mu_C = \operatorname{mult}\delta_C = \#H(v_1,v_2,v_4) = \operatorname{mult}\Delta_{[CD]} = \#H(v_1,v_2) = -\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} > 0,$$
$$\mu_2 = \operatorname{mult}\delta_2 = \#H(v_1,v_3) = \gcd\left(\Psi, \left\|\begin{matrix} a_1 & c_1 \\ a_3 & c_3 \end{matrix}\right\|, \left\|\begin{matrix} b_1 & c_1 \\ b_3 & c_3 \end{matrix}\right\|\right) > 0,$$

with $\Psi > 0$ as in (39).

Although we did not choose $\delta'_1 = \operatorname{cone}(v_0,v_1,v_2)$ to be part of a simplicial decomposition of Δ_D, we will consider its multiplicity as well. As in Case I, we then find that

$$\mu_1 = \operatorname{mult}\delta_1 = \#H(v_0,v_1,v_3) = \begin{vmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = \mu_A\mu_B\varphi_{AB} > 0 \qquad \text{and}$$

$$\mu_1' = \operatorname{mult}\delta_1' = \#H(v_0,v_1,v_2) = \begin{vmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \mu_B\mu_C\varphi_{BC} > 0.$$

Finally, we will derive a more useful formula for

$$\mu_3 = \operatorname{mult}\delta_3 = \#H(v_1,v_2,v_3) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} > 0,$$

similar to the ones for μ_1 and μ_1', in Subsection 4.17.

4.9. **The sums $\Sigma(\cdot)(s_0)$ and $\Sigma(\cdot)'(s_0)$.** Since the corresponding multiplicities equal one, we find that

$$\Sigma(\Delta_{[AB]})(s_0) = \Sigma(\Delta_{[BC]})(s_0) = \Sigma(\Delta_{\tau_0})(s_0) = \Sigma(\Delta_{\tau_1})(s_0) = 1.$$

From the overview of the multiplicities, it is also clear[24] that we may put

$$\begin{gathered} H_A = H(v_0,v_3,v_4) = H(v_0,v_3), \\ H_B = H(v_0,v_1,v_4) = H(v_0,v_1), \\ H_C = H(v_1,v_2,v_4) = H(v_1,v_2), \\ H_1 = H(v_0,v_1,v_3), \qquad H_2 = H(v_1,v_3), \qquad H_3 = H(v_1,v_2,v_3). \end{gathered}$$

It follows that

$$\begin{aligned} \Sigma_A &= \Sigma(\delta_A)(s_0) = \Sigma(\Delta_{[AD]})(s_0) = \sum\nolimits_{h\in H_A} p^{\sigma(h)+m(h)s_0}; \\ \Sigma_B &= \Sigma(\Delta_B)(s_0) = \Sigma(\Delta_{[BD]})(s_0) = \sum\nolimits_{h\in H_B} p^{\sigma(h)+m(h)s_0}; \\ \Sigma_C &= \Sigma(\delta_C)(s_0) = \Sigma(\Delta_{[CD]})(s_0) = \sum\nolimits_{h\in H_C} p^{\sigma(h)+m(h)s_0}; \\ \Sigma_i &= \Sigma(\delta_i)(s_0) = \sum\nolimits_{h\in H_i} p^{\sigma(h)+m(h)s_0}; \qquad i=1,2,3; \\ \Sigma_B' &= \Sigma(\Delta_B)'(s_0) = \Sigma(\Delta_{[BD]})'(s_0) = \frac{d}{ds}\left[\sum\nolimits_{h\in H_B} p^{\sigma(h)+m(h)s}\right]\Big|_{s=s_0}; \\ \Sigma_1' &= \Sigma(\delta_1)'(s_0) = \frac{d}{ds}\left[\sum\nolimits_{h\in H_1} p^{\sigma(h)+m(h)s}\right]\Big|_{s=s_0}. \end{aligned}$$

Let us for the rest of this section denote by w the vector

$$w = (1,1,1) + s_0(x_D,y_D,1) \in \mathbf{C}^3.$$

Then since $\overline{\Delta_D}$ contains all points of H_V; $V = A,B,C,1,2,3$; we have moreover that

$$\begin{aligned} &\Sigma_V = \sum\nolimits_{h\in H_V} p^{w\cdot h}; && V = A,B,C,1,2,3; \\ \text{and} \qquad &\Sigma_W' = (\log p)\sum\nolimits_{h\in H_W} m(h)p^{w\cdot h}; && W = B,1. \end{aligned}$$

[24]See Case I for more details.

4.10. **Simplified formulas for R_2 and R_1.** Let us put

$$F_2 = p^{w\cdot v_2} - 1 = p^{\sigma_2 + m_2 s_0} - 1 \qquad \text{and} \qquad F_3 = p^{w\cdot v_3} - 1 = p^{\sigma_3 + m_3 s_0} - 1.$$

Then, exploiting the information above on the numbers N_τ and the multiplicities of the cones, we obtain the following new formulas for R_2 and R_1:

$$R_2 = \left(\frac{p-1}{p}\right)^3 \left(\frac{\Sigma_B}{1-p^{-s_0-1}} + \frac{\Sigma_1}{F_3}\right), \tag{42}$$

$$\begin{aligned} R_1 = (\log p)\left(\frac{p-1}{p}\right)^3 \cdot \\ \left[\frac{1}{1-p^{-s_0-1}}\left(\frac{m_1\Sigma_A}{F_3} + \frac{\Sigma_B'}{\log p} - \frac{\Sigma_B}{p^{s_0+1}-1} + \frac{m_0\Sigma_C}{F_2} + m_0 + m_1\right)\right. \\ \left. + \frac{\Sigma_1'}{(\log p)F_3} - \frac{m_3(F_3+1)\Sigma_1}{F_3^2} + \frac{m_0\Sigma_2}{F_3} + \frac{m_0\Sigma_3}{F_2F_3}\right]. \end{aligned} \tag{43}$$

Note that the 'unknown' numbers N_0 and N_1 disappear from the equation.

4.11. **Vector identities.** We will quite often use the following identities:

$$\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} v_0 - \begin{vmatrix} a_0 & b_0 \\ a_3 & b_3 \end{vmatrix} v_1 + \begin{vmatrix} a_0 & b_0 \\ a_1 & b_1 \end{vmatrix} v_3 = \quad \Psi v_0 - \mu_A v_1 - \mu_B v_3 = (0,0,\mu_1), \tag{44}$$

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} v_0 - \begin{vmatrix} a_0 & b_0 \\ a_2 & b_2 \end{vmatrix} v_1 + \begin{vmatrix} a_0 & b_0 \\ a_1 & b_1 \end{vmatrix} v_2 = -\mu_C v_0 + \quad \Omega v_1 - \mu_B v_2 = (0,0,\mu_1'), \tag{45}$$

$$\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} v_1 - \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} v_2 + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} v_3 = \quad \Theta v_1 - \quad \Psi v_2 - \mu_C v_3 = (0,0,\mu_3). \tag{46}$$

Hereby $\Psi, \Omega, \Theta > 0$ are as introduced in (39). As also mentioned in Case I, these equations simply express the equalities of the last rows of the identical matrices $(\operatorname{adj} M)M$ and $(\det M)I$ for M the respective matrices

$$\begin{pmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{pmatrix}, \qquad \begin{pmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}, \qquad \text{and} \qquad \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

with respective determinants μ_1, μ_1', and μ_3.

Useful consequences of (44–46) arise from making the dot product with $w = (1,1,1) + s_0(x_D, y_D, 1)$ on all sides of the equations:

$$-\Psi(w\cdot v_0) + \mu_A(w\cdot v_1) + \mu_B(w\cdot v_3) = \mu_1(-s_0-1), \tag{47}$$

$$\mu_C(w\cdot v_0) - \quad \Omega(w\cdot v_1) + \mu_B(w\cdot v_2) = \mu_1'(-s_0-1), \tag{48}$$

$$-\Theta(w\cdot v_1) + \quad \Psi(w\cdot v_2) + \mu_C(w\cdot v_3) = \mu_3(-s_0-1). \tag{49}$$

4.12. **Points of H_A, H_B, H_C, H_2, H_1 and additional relations.** Based on the discussion on integral points in fundamental parallelepipeds in Section 1, we can state that the points of H_A, H_B, H_C, and H_2 are given by

$$\left\{\frac{i\xi_A}{\mu_A}\right\} v_0 + \frac{i}{\mu_A} v_3; \qquad i = 0, \ldots, \mu_A - 1; \tag{50}$$

by

$$\left\{\frac{j\xi_B}{\mu_B}\right\} v_0 + \frac{j}{\mu_B} v_1; \qquad j = 0, \ldots, \mu_B - 1; \tag{51}$$

by

$$\left\{\frac{i\xi_C}{\mu_C}\right\}v_1+\frac{i}{\mu_C}v_2;\qquad i=0,\dots,\mu_C-1;\tag{52}$$

and by

$$\left\{\frac{j\xi_2}{\mu_2}\right\}v_1+\frac{j}{\mu_2}v_3;\qquad j=0,\dots,\mu_2-1;\tag{53}$$

respectively. Here ξ_A denotes the unique element $\xi_A\in\{0,\dots,\mu_A-1\}$ such that $\xi_A v_0+v_3$ belongs to $\mu_A\mathbf{Z}^3$. It follows that ξ_A is coprime to μ_A. (Analogously for ξ_B,ξ_C, and ξ_2.)

In exactly the same way as we did in Case I for the points of H_C (cfr. Subsection 2.13), we obtain that the $\mu_1=\mu_A\mu_B\varphi_{AB}$ points of $H_1=H(v_0,v_1,v_3)$ are precisely

$$\begin{gathered}\left\{\frac{i\xi_A\mu_B\varphi_{AB}+j\xi_B\mu_A\varphi_{AB}-k\Psi}{\mu_1}\right\}v_0+\frac{j\varphi_{AB}+k}{\mu_B\varphi_{AB}}v_1+\frac{i\varphi_{AB}+k}{\mu_A\varphi_{AB}}v_3;\\ i=0,\dots,\mu_A-1;\quad j=0,\dots,\mu_B-1;\quad k=0,\dots,\varphi_{AB}-1.\end{gathered}\tag{54}$$

On the other hand, we also know from Section 1 that $\mu_2\mid\mu_1$ and that when h runs through the elements of H_1, its v_0-coordinate h_0 runs precisely μ_2 times through the numbers

$$\frac{l\mu_2}{\mu_1};\qquad l=0,\dots,\frac{\mu_1}{\mu_2}-1.$$

This implies that[25]

$$\big\{\{i\mu_B\varphi_{AB}+j\mu_A\varphi_{AB}-k\Psi\}_{\mu_1}\big\}_{i,j,k=0}^{\mu_A-1,\mu_B-1,\varphi_{AB}-1}=\big\{l\mu_2\big\}_{l=0}^{\mu_1/\mu_2-1}.$$

From this equality of sets, we easily conclude that

$$\mu_2\mid\mu_A\varphi_{AB},\mu_B\varphi_{AB},\Psi,$$

what we already knew[26], but also that μ_2 can be written as a linear combination with integer coefficients of $\mu_A\varphi_{AB},\mu_B\varphi_{AB}$, and Ψ. Hence

$$\mu_2=\gcd(\mu_A\varphi_{AB},\mu_B\varphi_{AB},\Psi).$$

Recall from (44) that

$$\Psi v_0-\mu_A v_1-\mu_B v_3=(0,0,\mu_1).$$

If we put $\gamma=\gcd(\mu_A\varphi_{AB},\Psi)$, we have $\gamma\mid\mu_1$, and therefore it follows that

$$\mu_B\varphi_{AB}v_3=\Psi\varphi_{AB}v_0-\mu_A\varphi_{AB}v_1-(0,0,\varphi_{AB}\mu_1)\in\gamma\mathbf{Z}^3.$$

---

[25]First, note that in the left-hand side of the equation, the inner curly brackets denote the reduction of the argument modulo μ_1 (cfr. Notation 1.6), while the outer curly brackets serve as set delimiters. Secondly, recall that the maps $i\mapsto\{i\xi_A\}_{\mu_A}$ and $j\mapsto\{j\xi_B\}_{\mu_B}$ are permutations of $\{0,\dots,\mu_A-1\}$ and $\{0,\dots,\mu_B-1\}$, respectively, so that after reordering the elements of the set, we can indeed omit the ξ_A and ξ_B from the equation.

[26]The fact that $\mu_2\mid\mu_A\varphi_{AB},\mu_B\varphi_{AB}$ was shown in several ways in the proof of Theorem 1.7(v), while it follows from Proposition 1.4 that $\mu_2=\gcd\left(\Psi,\left\|\begin{smallmatrix}a_1&c_1\\a_3&c_3\end{smallmatrix}\right\|,\left\|\begin{smallmatrix}b_1&c_1\\b_3&c_3\end{smallmatrix}\right\|\right)$.

The primitivity of v_3 now implies that $\gamma \mid \mu_B\varphi_{AB}$, and thus we obtain that

$$\begin{aligned}\mu_2 &= \gcd(\mu_A\varphi_{AB}, \mu_B\varphi_{AB}, \Psi)\\ &= \gcd(\mu_A\varphi_{AB}, \Psi)\\ &= \gcd(\mu_B\varphi_{AB}, \Psi),\end{aligned}$$

the last equality due to the symmetry of the argument above.

Finally, let us denote

$$\frac{\mu_A\varphi_{AB}}{\mu_2} = \frac{\mu_1}{\mu_B\mu_2} = \varphi_{B2} \in \mathbf{Z}_{>0}, \qquad \frac{\mu_B\varphi_{AB}}{\mu_2} = \frac{\mu_1}{\mu_A\mu_2} = \varphi_{A2} \in \mathbf{Z}_{>0}, \quad \text{and} \quad \frac{\Psi}{\mu_2} = \psi \in \mathbf{Z}_{>0}, \tag{55}$$

resulting in

$$\mu_1 = \mu_A\mu_B\varphi_{AB} = \mu_A\mu_2\varphi_{A2} = \mu_B\mu_2\varphi_{B2}.$$

4.13. Investigation of the $\Sigma_\bullet$ and the $\Sigma'_\bullet$, except for Σ'_1, Σ_3.

4.13.1. *The sum* Σ_B. Because in this case τ_0 and τ_1 both contribute to s_0, and therefore $p^{w\cdot v_0} = p^{w\cdot v_1} = 1$, the term Σ_B plays a special role. By (51) and the fact that $p^{a(w\cdot v_0)} = p^{\{a\}(w\cdot v_0)}$ for every $a \in \mathbf{R}$, we have

$$\Sigma_B = \sum_{h\in H_B} p^{w\cdot h} = \sum_{j=0}^{\mu_B-1}\left(p^{\frac{\xi_B(w\cdot v_0)+w\cdot v_1}{\mu_B}}\right)^j = \sum_{j=0}^{\mu_B-1}\left(p^{w\cdot h^*}\right)^j,$$

with

$$h^* = \frac{\xi_B}{\mu_B}v_0 + \frac{1}{\mu_B}v_1 \in \mathbf{Z}^3,$$

a generating element of H_B (if $\mu_B > 1$).

Unlike, e.g., $p^{(\xi_A(w\cdot v_0)+w\cdot v_3)/\mu_A}$, appearing in Formula (58) for Σ_A below, the μ_Bth root of unity $p^{w\cdot h^*} = p^{(\xi_B(w\cdot v_0)+w\cdot v_1)/\mu_B}$ may equal one, but may as well differ from one. We need to distinguish between these two cases. As

$$s_0 = -\frac{\sigma_0}{m_0} + \frac{2n\pi i}{\gcd(m_0, m_1)\log p} = -\frac{\sigma_1}{m_1} + \frac{2n\pi i}{\gcd(m_0, m_1)\log p}$$

for some $n \in \mathbf{Z}$ and hence

$$p^{w\cdot h^*} = p^{\sigma(h^*)+m(h^*)s_0} = \exp\frac{2nm(h^*)\pi i}{\gcd(m_0, m_1)},$$

we see that $p^{w\cdot h^*} = 1$ if and only if

$$n^* = \frac{\gcd(m_0, m_1)}{\gcd(m_0, m_1, m(h^*))} \mid n.$$

In this way we find

$$\Sigma_B = \sum_{j=0}^{\mu_B-1}\left(p^{w\cdot h^*}\right)^j = \begin{cases}\sum_j 1 = \mu_B, & \text{if } n^* \mid n;\\[4pt] \dfrac{\left(p^{w\cdot h^*}\right)^{\mu_B} - 1}{p^{w\cdot h^*} - 1} = 0, & \text{otherwise.}\end{cases} \tag{56}$$

Let us next look at Σ'_B.

4.13.2. *The sum Σ'_B.* As we know, the μ_B points of H_B are given by

$$\left\{\frac{j\xi_B}{\mu_B}\right\} v_0 + \frac{j}{\mu_B} v_1; \qquad j = 0, \ldots, \mu_B - 1;$$

but if ξ'_B denotes the unique element $\xi'_B \in \{0, \ldots, \mu_B - 1\}$ such that $\xi_B \xi'_B \equiv 1 \mod \mu_B$, they are as well given by

$$\frac{j}{\mu_B} v_0 + \left\{\frac{j\xi'_B}{\mu_B}\right\} v_1; \qquad j = 0, \ldots, \mu_B - 1.$$

Recall that we introduced Σ'_B as

$$\begin{aligned} \Sigma'_B &= \Sigma(\Delta_B)'(s_0) = \Sigma(\Delta_{[BD]})'(s_0) \\ &= \frac{d}{ds}\left[\sum_{h \in H_B} p^{\sigma(h) + m(h)s}\right]\Bigg|_{s=s_0} \\ &= (\log p) \sum_{h \in H_B} m(h) p^{\sigma(h) + m(h)s_0} \\ &= (\log p) \sum_h m(h) p^{w \cdot h}. \end{aligned}$$

Hence if we write $h = h_0 v_0 + h_1 v_1$ for $h \in H_B = H(v_0, v_1)$, we find

$$\begin{aligned} \frac{\Sigma'_B}{\log p} &= m_0 \sum_{h \in H_B} h_0 p^{h_0(w\cdot v_0) + h_1(w \cdot v_1)} + m_1 \sum_{h \in H_B} h_1 p^{h_0(w\cdot v_0) + h_1(w \cdot v_1)} \\ &= \frac{m_0}{\mu_B} \sum_{j=0}^{\mu_B - 1} j \Big(p^{\frac{w\cdot v_0 + \xi'_B(w\cdot v_1)}{\mu_B}}\Big)^j + \frac{m_1}{\mu_B} \sum_{j=0}^{\mu_B - 1} j \Big(p^{\frac{\xi_B(w\cdot v_0) + w\cdot v_1}{\mu_B}}\Big)^j. \end{aligned}$$

As

$$p^{\frac{w\cdot v_0 + \xi'_B(w\cdot v_1)}{\mu_B}} = \Big(p^{\frac{\xi_B(w\cdot v_0) + w\cdot v_1}{\mu_B}}\Big)^{\xi'_B} \quad \text{and} \quad p^{\frac{\xi_B(w\cdot v_0) + w\cdot v_1}{\mu_B}} = \Big(p^{\frac{w\cdot v_0 + \xi'_B(w\cdot v_1)}{\mu_B}}\Big)^{\xi_B}, \tag{57}$$

the numbers $p^{(\xi_B(w\cdot v_0) + w\cdot v_1)/\mu_B}$ and $p^{(w\cdot v_0 + \xi'_B(w\cdot v_1))/\mu_B}$ are either both one (if $n^* \mid n$) or both different from one (if $n^* \nmid n$). We obtain

$$\frac{\Sigma'_B}{\log p} = \begin{cases} \dfrac{m_0 + m_1}{\mu_B} \displaystyle\sum_{j=0}^{\mu_B - 1} j = \dfrac{(m_0 + m_1)(\mu_B - 1)}{2}, & \text{if } n^* \mid n; \\ \dfrac{m_0}{p^{\frac{w\cdot v_0 + \xi'_B(w\cdot v_1)}{\mu_B}} - 1} + \dfrac{m_1}{p^{\frac{\xi_B(w\cdot v_0) + w\cdot v_1}{\mu_B}} - 1}, & \text{otherwise.} \end{cases}$$

4.13.3. *The sums Σ_A, Σ_C, and Σ_2.* From (50, 52, 53) and the fact that $p^{w\cdot v_0} = p^{w\cdot v_1} = 1$, we obtain in the same way as in Case I that

$$(58)\qquad \begin{aligned} \Sigma_A &= \sum_{h\in H_A} p^{w\cdot h} = \sum_{i=0}^{\mu_A-1}\left(p^{\frac{\xi_A(w\cdot v_0)+w\cdot v_3}{\mu_A}}\right)^i = \frac{F_3}{p^{\frac{\xi_A(w\cdot v_0)+w\cdot v_3}{\mu_A}}-1},\\ \Sigma_C &= \sum_{h\in H_C} p^{w\cdot h} = \sum_{i=0}^{\mu_C-1}\left(p^{\frac{\xi_C(w\cdot v_1)+w\cdot v_2}{\mu_C}}\right)^i = \frac{F_2}{p^{\frac{\xi_C(w\cdot v_1)+w\cdot v_2}{\mu_C}}-1}, \qquad \text{and}\\ \Sigma_2 &= \sum_{h\in H_2} p^{w\cdot h} = \sum_{j=0}^{\mu_2-1}\left(p^{\frac{\xi_2(w\cdot v_1)+w\cdot v_3}{\mu_2}}\right)^j = \frac{F_3}{p^{\frac{\xi_2(w\cdot v_1)+w\cdot v_3}{\mu_2}}-1}. \end{aligned}$$

In Case I we also observed that $\xi_A v_0 + v_3 \in \mu_A\mathbf{Z}^3$, $\xi_B v_0 + v_1 \in \mu_B\mathbf{Z}^3$, and (44) give rise to $(\xi_A\mu_B + \xi_B\mu_A + \Psi)v_0 \in \mu_A\mu_B\mathbf{Z}^3$ and hence to

$$(59)\qquad \frac{\xi_A\mu_B + \xi_B\mu_A + \Psi}{\mu_A\mu_B} \in \mathbf{Z}.$$

Using (47) it follows that

$$(60)\qquad \begin{aligned} p^{\frac{\xi_A(w\cdot v_0)+w\cdot v_3}{\mu_A}}\, p^{\frac{\xi_B(w\cdot v_0)+w\cdot v_1}{\mu_B}} &= p^{\frac{(\xi_A\mu_B+\xi_B\mu_A)(w\cdot v_0)+\mu_A(w\cdot v_1)+\mu_B(w\cdot v_3)}{\mu_A\mu_B}}\\ &= p^{\frac{-\Psi(w\cdot v_0)+\mu_A(w\cdot v_1)+\mu_B(w\cdot v_3)}{\mu_A\mu_B}} = p^{\varphi_{AB}(-s_0-1)}. \end{aligned}$$

Analogously, $v_0 + \xi'_B v_1 \in \mu_B\mathbf{Z}^3$, $\xi_C v_1 + v_2 \in \mu_C\mathbf{Z}^3$, (45), and (48) yield

$$(61)\qquad \begin{aligned} p^{\frac{w\cdot v_0+\xi'_B(w\cdot v_1)}{\mu_B}}\, p^{\frac{\xi_C(w\cdot v_1)+w\cdot v_2}{\mu_C}} &= p^{\frac{\mu_C(w\cdot v_0)+(\xi'_B\mu_C+\xi_C\mu_B)(w\cdot v_1)+\mu_B(w\cdot v_2)}{\mu_B\mu_C}}\\ &= p^{\frac{\mu_C(w\cdot v_0)-\Omega(w\cdot v_1)+\mu_B(w\cdot v_2)}{\mu_B\mu_C}} = p^{\varphi_{BC}(-s_0-1)}, \end{aligned}$$

while $v_0 + \xi'_B v_1 \in \mu_B\mathbf{Z}^3$, $\xi_2 v_1 + v_3 \in \mu_2\mathbf{Z}^3$, (44), (47), and (55) lead to

$$(62)\qquad \begin{aligned} \left(p^{\frac{w\cdot v_0+\xi'_B(w\cdot v_1)}{\mu_B}}\right)^{-\psi} p^{\frac{\xi_2(w\cdot v_1)+w\cdot v_3}{\mu_2}} &= p^{\frac{-\Psi(w\cdot v_0)+(-\xi'_B\Psi+\xi_2\mu_B)(w\cdot v_1)+\mu_B(w\cdot v_3)}{\mu_B\mu_2}}\\ &= p^{\frac{-\Psi(w\cdot v_0)+\mu_A(w\cdot v_1)+\mu_B(w\cdot v_3)}{\mu_B\mu_2}} = p^{\varphi_{B2}(-s_0-1)}. \end{aligned}$$

Consequently, if $n^* \mid n$ and hence $p^{(\xi_B(w\cdot v_0)+w\cdot v_1)/\mu_B} = p^{(w\cdot v_0+\xi'_B(w\cdot v_1))/\mu_B} = 1$, one has that

$$(63)\qquad \begin{aligned} &\Sigma_A = \frac{F_3}{p^{\varphi_{AB}(-s_0-1)}-1}, \qquad \Sigma_C = \frac{F_2}{p^{\varphi_{BC}(-s_0-1)}-1},\\ &\text{and}\qquad \Sigma_2 = \frac{F_3}{p^{\varphi_{B2}(-s_0-1)}-1}. \end{aligned}$$

4.13.4. *The sum Σ_1.* If for $h \in H_1 = H(v_0, v_1, v_3)$, we denote by (h_0, h_1, h_3) the coordinates of h with respect to the basis (v_0, v_1, v_3), then by (54) and $p^{w\cdot v_0} = 1$

we have that

$$\begin{aligned}\Sigma_1 &= \sum_{h\in H_1} p^{w\cdot h}\\ &= \sum_{h} p^{h_0(w\cdot v_0)+h_1(w\cdot v_1)+h_3(w\cdot v_3)}\\ &= \sum_{i=0}^{\mu_A-1}\sum_{j=0}^{\mu_B-1}\sum_{k=0}^{\varphi_{AB}-1} p^{\frac{i\xi_A\mu_B\varphi_{AB}+j\xi_B\mu_A\varphi_{AB}-k\Psi}{\mu_1}(w\cdot v_0)+\frac{j\varphi_{AB}+k}{\mu_B\varphi_{AB}}(w\cdot v_1)+\frac{i\varphi_{AB}+k}{\mu_A\varphi_{AB}}(w\cdot v_3)}\\ &= \sum_i\left(p^{\frac{\xi_A(w\cdot v_0)+w\cdot v_3}{\mu_A}}\right)^i \sum_j\left(p^{\frac{\xi_B(w\cdot v_0)+w\cdot v_1}{\mu_B}}\right)^j \sum_k\left(p^{\frac{-\Psi(w\cdot v_0)+\mu_A(w\cdot v_1)+\mu_B(w\cdot v_3)}{\mu_1}}\right)^k\\ &= \Sigma_A\Sigma_B\,\frac{p^{\varphi_{AB}(-s_0-1)}-1}{p^{-s_0-1}-1},\end{aligned}$$

where in the last step we again used (47). It now follows from (56) and (63) that

$$\Sigma_1 = \begin{cases}\dfrac{\mu_B F_3}{p^{-s_0-1}-1}, & \text{if } n^*\mid n;\\ 0, & \text{otherwise.}\end{cases}$$

4.14. **Proof of $R_2 = 0$ and a new formula for R_1.** If we fill in the formulas for Σ_B and Σ_1 in Formula (42) for R_2, we obtain

$$\begin{aligned}R_2 &= \left(\frac{p-1}{p}\right)^3\left(\frac{\Sigma_B}{1-p^{-s_0-1}}+\frac{\Sigma_1}{F_3}\right)\\ &= \left(\frac{p-1}{p}\right)^3\left(\frac{\mu_B}{1-p^{-s_0-1}}+\frac{\mu_B}{p^{-s_0-1}-1}\right)\\ &= 0\end{aligned}$$

in the case that $n^*\mid n$ and clearly the same result in the other case as well.

Let us check how much progress we made on R_1. First of all, denote by R_1' the third factor in Formula (43) for R_1; i.e., put

$$R_1 = (\log p)\left(\frac{p-1}{p}\right)^3 R_1'. \tag{64}$$

Obviously, we want to prove that $R_1' = 0$. Secondly, let us from now on denote p^{-s_0-1} by q.

If we then fill in the formulas for $\Sigma_A, \Sigma_B, \Sigma_C, \Sigma_1, \Sigma_2$, and Σ_B' obtained above in the formula for R_1', we find that in the case $n^*\mid n$, the 'residue' R_1' equals

$$\begin{aligned}R_1' = \frac{1}{1-q}&\left(\frac{m_0}{1-q^{-\varphi_{BC}}}+\frac{m_1}{1-q^{-\varphi_{AB}}}+\frac{\mu_B}{1-q^{-1}}+\frac{m_3\mu_B(F_3+1)}{F_3}\right.\\ &\left.+\frac{(m_0+m_1)(\mu_B-1)}{2}\right)+\frac{\Sigma_1'}{(\log p)F_3}+\frac{m_0}{q^{\varphi_{B2}}-1}+\frac{m_0\Sigma_3}{F_2F_3}.\end{aligned} \tag{65}$$

In the complementary case we have

$$
\begin{aligned}
R_1' = \frac{m_1}{1-q}&\left(\frac{1}{p^{\frac{\xi_A(w\cdot v_0)+w\cdot v_3}{\mu_A}}-1}+\frac{1}{p^{\frac{\xi_B(w\cdot v_0)+w\cdot v_1}{\mu_B}}-1}+1\right)\\
&\qquad+\frac{m_0}{1-q}\left(\frac{1}{p^{\frac{w\cdot v_0+\xi_B'(w\cdot v_1)}{\mu_B}}-1}+\frac{1}{p^{\frac{\xi_C(w\cdot v_1)+w\cdot v_2}{\mu_C}}-1}+1\right)\\
&\qquad\qquad+\frac{\Sigma_1'}{(\log p)F_3}+\frac{m_0}{p^{\frac{\xi_2(w\cdot v_1)+w\cdot v_3}{\mu_2}}-1}+\frac{m_0\Sigma_3}{F_2F_3},
\end{aligned}
$$

and by (60–61) this is,

$$
\begin{aligned}
R_1' = \frac{m_1}{1-q}&\frac{q^{\varphi_{AB}}-1}{\left(p^{\frac{\xi_A(w\cdot v_0)+w\cdot v_3}{\mu_A}}-1\right)\left(p^{\frac{\xi_B(w\cdot v_0)+w\cdot v_1}{\mu_B}}-1\right)}\\
&\qquad+\frac{m_0}{1-q}\frac{q^{\varphi_{BC}}-1}{\left(p^{\frac{w\cdot v_0+\xi_B'(w\cdot v_1)}{\mu_B}}-1\right)\left(p^{\frac{\xi_C(w\cdot v_1)+w\cdot v_2}{\mu_C}}-1\right)}\\
&\qquad\qquad+\frac{\Sigma_1'}{(\log p)F_3}+\frac{m_0}{p^{\frac{\xi_2(w\cdot v_1)+w\cdot v_3}{\mu_2}}-1}+\frac{m_0\Sigma_3}{F_2F_3}.
\end{aligned}
\tag{66}
$$

4.15. **Study of Σ_1'.** The term Σ_1' was defined as

$$
\begin{aligned}
\Sigma_1' &= \Sigma(\delta_1)'(s_0)\\
&= \frac{d}{ds}\left[\sum_{h\in H_1} p^{\sigma(h)+m(h)s}\right]\Bigg|_{s=s_0}\\
&= (\log p)\sum_h m(h)p^{\sigma(h)+m(h)s_0}\\
&= (\log p)\sum_h m(h)p^{w\cdot h}.
\end{aligned}
$$

Writing $h = h_0v_0 + h_1v_1 + h_3v_3$ for $h \in H_1 = H(v_0, v_1, v_3)$, we have that

$$
\begin{aligned}
\frac{\Sigma_1'}{\log p} &= \sum_{h\in H_1}(h_0m_0+h_1m_1+h_3m_3)p^{w\cdot h}\\
&= m_0\Sigma_1^{(0)} + m_1\Sigma_1^{(1)} + m_3\Sigma_1^{(3)},
\end{aligned}
\tag{67}
$$

with

$$
\Sigma_1^{(i)} = \sum_{h\in H_1} h_i p^{w\cdot h}; \qquad i = 0, 1, 3.
$$

We will now calculate $\Sigma_1^{(1)}, \Sigma_1^{(3)}$, and $\Sigma_1^{(0)}$.

4.15.1. *The sum* $\Sigma_1^{(1)}$. With the notation $q = p^{-s_0-1}$, Identity (47) yields

$$
p^{\frac{-\Psi(w\cdot v_0)+\mu_A(w\cdot v_1)+\mu_B(w\cdot v_3)}{\mu_1}} = p^{-s_0-1} = q.
$$

Hence, based on the description (54) of the points of H_1 and proceeding as in Paragraph 4.13.4, we obtain

$$
\begin{aligned}
\Sigma_1^{(1)} &= \sum_{h\in H_1} h_1 p^{w\cdot h}\\
&= \sum_{i=0}^{\mu_A-1}\sum_{j=0}^{\mu_B-1}\sum_{k=0}^{\varphi_{AB}-1} \frac{j\varphi_{AB}+k}{\mu_B\varphi_{AB}} \Big(p^{\frac{\xi_A(w\cdot v_0)+w\cdot v_3}{\mu_A}}\Big)^i \Big(p^{\frac{\xi_B(w\cdot v_0)+w\cdot v_1}{\mu_B}}\Big)^j q^k\\
&= \frac{\Sigma_A}{\mu_B}\left[\sum_j j\Big(p^{\frac{\xi_B(w\cdot v_0)+w\cdot v_1}{\mu_B}}\Big)^j \sum_k q^k + \frac{1}{\varphi_{AB}}\sum_j \Big(p^{\frac{\xi_B(w\cdot v_0)+w\cdot v_1}{\mu_B}}\Big)^j \sum_k kq^k\right]. \qquad (68)
\end{aligned}
$$

For $n^* \mid n$, by Formula (63) for Σ_A, we then find

$$
\begin{aligned}
\Sigma_1^{(1)} &= \frac{F_3}{\mu_B(q^{\varphi_{AB}}-1)}\bigg(\frac{\mu_B(\mu_B-1)}{2}\frac{q^{\varphi_{AB}}-1}{q-1}\\
&\qquad\qquad + \frac{1}{\varphi_{AB}}\,\mu_B\,\frac{q^{\varphi_{AB}}(\varphi_{AB}q-\varphi_{AB}-q)+q}{(q-1)^2}\bigg)\\
&= \frac{F_3}{1-q}\left(-\frac{\mu_B-1}{2} - \frac{1}{1-q^{-\varphi_{AB}}} + \frac{1}{\varphi_{AB}(1-q^{-1})}\right), \qquad (69)
\end{aligned}
$$

while in the complementary case, we conclude

$$
\begin{aligned}
\Sigma_1^{(1)} &= \frac{F_3}{\mu_B\Big(p^{\frac{\xi_A(w\cdot v_0)+w\cdot v_3}{\mu_A}}-1\Big)}\,\frac{\mu_B}{p^{\frac{\xi_B(w\cdot v_0)+w\cdot v_1}{\mu_B}}-1}\,\frac{q^{\varphi_{AB}}-1}{q-1}\\
&= \frac{F_3}{q-1}\,\frac{q^{\varphi_{AB}}-1}{\Big(p^{\frac{\xi_A(w\cdot v_0)+w\cdot v_3}{\mu_A}}-1\Big)\Big(p^{\frac{\xi_B(w\cdot v_0)+w\cdot v_1}{\mu_B}}-1\Big)}. \qquad (70)
\end{aligned}
$$

Note that if $n^* \nmid n$, the second term of (68) vanishes, as the sum over j equals zero in this case.

4.15.2. *The sum* $\Sigma_1^{(3)}$. Similarly, $\Sigma_1^{(3)}$ is given by

$$
\begin{aligned}
\Sigma_1^{(3)} &= \sum_{h\in H_1} h_3 p^{w\cdot h}\\
&= \sum_{i=0}^{\mu_A-1}\sum_{j=0}^{\mu_B-1}\sum_{k=0}^{\varphi_{AB}-1} \frac{i\varphi_{AB}+k}{\mu_A\varphi_{AB}} \Big(p^{\frac{\xi_A(w\cdot v_0)+w\cdot v_3}{\mu_A}}\Big)^i \Big(p^{\frac{\xi_B(w\cdot v_0)+w\cdot v_1}{\mu_B}}\Big)^j q^k.
\end{aligned}
$$

If $n^* \nmid n$, the sum over j vanishes and $\Sigma_1^{(3)} = 0$. In the other case, one has

$$
p^{\frac{\xi_B(w\cdot v_0)+w\cdot v_1}{\mu_B}} = 1 \qquad \text{and subsequently} \qquad p^{\frac{\xi_A(w\cdot v_0)+w\cdot v_3}{\mu_A}} = p^{\varphi_{AB}(-s_0-1)} = q^{\varphi_{AB}},
$$

leading to

$$
\begin{aligned}
\Sigma_1^{(3)} &= \frac{\mu_B}{\mu_A\varphi_{AB}} \sum_{i=0}^{\mu_A-1} \sum_{k=0}^{\varphi_{AB}-1} (i\varphi_{AB}+k) q^{i\varphi_{AB}+k} \\
&= \frac{\mu_B}{\mu_A\varphi_{AB}} \sum_{l=0}^{\mu_A\varphi_{AB}-1} l q^l \\
&= \frac{\mu_B}{\mu_A\varphi_{AB}} \frac{q^{\mu_A\varphi_{AB}}(\mu_A\varphi_{AB}q - \mu_A\varphi_{AB} - q) + q}{(q-1)^2} \\
&= \frac{\mu_B}{1-q}\left(-(F_3+1) + \frac{F_3}{\mu_A\varphi_{AB}(1-q^{-1})}\right). \qquad (71)
\end{aligned}
$$

Note that $q^{\mu_A\varphi_{AB}} = p^{\xi_A(w\cdot v_0)+w\cdot v_3} = p^{w\cdot v_3} = F_3+1$ in this case.

Let us now look at $\Sigma_1^{(0)}$.

4.15.3. *The sum* $\Sigma_1^{(0)}$. Still based on (54), this time we ought to consider the following sum:

$$
\begin{aligned}
\Sigma_1^{(0)} &= \sum_{h\in H_1} h_0 p^{w\cdot h} \\
&= \sum_{i=0}^{\mu_A-1} \sum_{j=0}^{\mu_B-1} \sum_{k=0}^{\varphi_{AB}-1} \left\{ \frac{i\xi_A\mu_B\varphi_{AB} + j\xi_B\mu_A\varphi_{AB} - k\Psi}{\mu_1} \right\} \cdot \\
&\qquad \left(p^{\frac{\xi_A(w\cdot v_0)+w\cdot v_3}{\mu_A}}\right)^i \left(p^{\frac{\xi_B(w\cdot v_0)+w\cdot v_1}{\mu_B}}\right)^j q^k.
\end{aligned}
$$

If we put

$$
j_0 = \left\lfloor \frac{i\xi_A\mu_B\varphi_{AB} - k\Psi}{\mu_A\varphi_{AB}} \right\rfloor, \qquad (72)
$$

we can write this sum as

$$
\Sigma_1^{(0)} = \frac{1}{\mu_B} \sum_{i=0}^{\mu_A-1} \sum_{k=0}^{\varphi_{AB}-1} \left(p^{\frac{\xi_A(w\cdot v_0)+w\cdot v_3}{\mu_A}}\right)^i q^k S(i,k), \qquad \text{with} \qquad (73)
$$

$$
S(i,k) = \sum_{j=0}^{\mu_B-1} \left(\left\{ \frac{i\xi_A\mu_B\varphi_{AB} - k\Psi}{\mu_A\varphi_{AB}} \right\} + \{j_0 + j\xi_B\}_{\mu_B} \right) \left(p^{\frac{\xi_B(w\cdot v_0)+w\cdot v_1}{\mu_B}}\right)^j.
$$

Since $\gcd(\xi_B,\mu_B)=1$, the map

$$
\{0,\ldots,\mu_B-1\} \to \{0,\ldots,\mu_B-1\} : j \mapsto \{j_0 + j\xi_B\}_{\mu_B}
$$

is a permutation, and with ξ_B' as before the unique element $\xi_B' \in \{0,\ldots,\mu_B-1\}$ such that $\xi_B\xi_B' \equiv 1 \bmod \mu_B$, the inverse permutation is given by

$$
\{0,\ldots,\mu_B-1\} \to \{0,\ldots,\mu_B-1\} : j \mapsto \{(j-j_0)\xi_B'\}_{\mu_B}.
$$

Therefore, after reordering the terms, the sum $S(i,k)$ can be written as

$$S(i,k) = \sum_{j=0}^{\mu_B-1} \left(\left\{ \frac{i\xi_A\mu_B\varphi_{AB} - k\Psi}{\mu_A\varphi_{AB}} \right\} + j \right) \left(p^{\frac{\xi_B(w\cdot v_0)+w\cdot v_1}{\mu_B}} \right)^{\{(j-j_0)\xi'_B\}_{\mu_B}} \tag{74}$$

$$= \left(p^{\frac{w\cdot v_0+\xi'_B(w\cdot v_1)}{\mu_B}} \right)^{-j_0} \sum_j \left(\left\{ \frac{i\xi_A\mu_B\varphi_{AB} - k\Psi}{\mu_A\varphi_{AB}} \right\} + j \right) \left(p^{\frac{w\cdot v_0+\xi'_B(w\cdot v_1)}{\mu_B}} \right)^{j}. \tag{75}$$

Indeed, as $p^{(\xi_B(w\cdot v_0)+w\cdot v_1)/\mu_B}$ is a μ_Bth root of unity, one may omit the curly brackets $\{\cdot\}_{\mu_B}$ in the exponent in (74). Expression (75) is then obtained by (57) and the fact that j_0 is independent of j.

It is now a good time to make a case distinction between $n^* \mid n$ and $n^* \nmid n$.

Case $n^* \mid n$. Since in this case

$$p^{\frac{\xi_A(w\cdot v_0)+w\cdot v_3}{\mu_A}} = p^{\varphi_{AB}(-s_0-1)} = q^{\varphi_{AB}} \qquad \text{and} \qquad p^{\frac{w\cdot v_0+\xi'_B(w\cdot v_1)}{\mu_B}} = 1,$$

Equations (73) and (75) give rise to

$$\Sigma_1^{(0)} = \frac{1}{\mu_B} \sum_{i=0}^{\mu_A-1} \sum_{k=0}^{\varphi_{AB}-1} q^{i\varphi_{AB}+k} \sum_{j=0}^{\mu_B-1} \left(\left\{ \frac{i\xi_A\mu_B\varphi_{AB} - k\Psi}{\mu_A\varphi_{AB}} \right\} + j \right)$$

$$= \frac{\mu_B-1}{2} \sum_{l=0}^{\mu_A\varphi_{AB}-1} q^l + \sum_i \sum_k \left\{ \frac{i\xi_A\mu_B\varphi_{AB} - k\Psi}{\mu_A\varphi_{AB}} \right\} q^{i\varphi_{AB}+k} \tag{76}$$

$$= -\frac{\mu_B-1}{2} \frac{F_3}{1-q} + T_2. \tag{77}$$

The second term of (76), which we temporarily denote by T_2, can be further simplified as follows. Either directly from $\xi_A v_0 + v_3 \in \mu_A \mathbf{Z}^3$ and (44), or as a corollary of (59), we have that

$$\frac{\xi_A\mu_B + \Psi}{\mu_A} \in \mathbf{Z}.$$

This makes that we can replace $\xi_A\mu_B$ by $-\Psi$ in the second term T_2 of (76):

$$\begin{aligned} T_2 &= \sum_{i=0}^{\mu_A-1} \sum_{k=0}^{\varphi_{AB}-1} \left\{ \frac{i\xi_A\mu_B\varphi_{AB} - k\Psi}{\mu_A\varphi_{AB}} \right\} q^{i\varphi_{AB}+k} \\ &= \sum_i \sum_k \left\{ \frac{i(-\Psi)\varphi_{AB} - k\Psi}{\mu_A\varphi_{AB}} \right\} q^{i\varphi_{AB}+k} \\ &= \sum_i \sum_k \left\{ \frac{-(i\varphi_{AB}+k)\Psi}{\mu_A\varphi_{AB}} \right\} q^{i\varphi_{AB}+k} \\ &= \sum_{l=0}^{\mu_A\varphi_{AB}-1} \left\{ \frac{-l\Psi}{\mu_A\varphi_{AB}} \right\} q^l. \end{aligned}$$

In Subsection 4.12 we showed that $\gcd(\mu_A\varphi_{AB}, \Psi) = \mu_2$. Therefore, if we recall that $\Psi = \psi\mu_2$ and $\mu_A\varphi_{AB} = \mu_2\varphi_{B2}$, we can write the fraction $-\Psi/\mu_A\varphi_{AB}$ in lowest terms and continue:

$$\begin{aligned}
T_2 &= \sum_{l=0}^{\mu_A\varphi_{AB}-1} \left\{ \frac{-l\psi}{\varphi_{B2}} \right\} q^l \\
&= \sum_{\iota=0}^{\mu_2-1} \sum_{\kappa=0}^{\varphi_{B2}-1} \left\{ \frac{-(\iota\varphi_{B2}+\kappa)\psi}{\varphi_{B2}} \right\} q^{\iota\varphi_{B2}+\kappa} \\
&= \sum_{\iota} (q^{\varphi_{B2}})^{\iota} \sum_{\kappa} \left\{ \frac{-\kappa\psi}{\varphi_{B2}} \right\} q^{\kappa} \\
&= \frac{F_3}{q^{\varphi_{B2}}-1} \sum_{\kappa} \left\{ \frac{-\kappa\overline{\psi}}{\varphi_{B2}} \right\} q^{\kappa},
\end{aligned}$$

where $\overline{\psi} = \{\psi\}_{\varphi_{B2}}$ denotes the reduction of ψ modulo φ_{B2}. Obviously, we have $\overline{\psi} \in \{0,\ldots,\varphi_{B2}-1\}$ and still $\gcd(\overline{\psi},\varphi_{B2}) = 1$. Note also that $\overline{\psi} = 0$ if and only if $\varphi_{B2} = 1$, and that if $\varphi_{B2} = 1$, then $T_2 = 0$. In what follows, we study T_2 under the assumption that $\varphi_{B2} > 1$.

For any real number a, one has that $\{-a\} = 0$ if $a \in \mathbf{Z}$, and $\{-a\} = 1 - \{a\}$ otherwise. Since $\overline{\psi}$ and φ_{B2} are coprime, the only $\kappa \in \{0,\ldots,\varphi_{B2}-1\}$ for which

$$\frac{\kappa\overline{\psi}}{\varphi_{B2}} \in \mathbf{Z},$$

is $\kappa = 0$. Consequently,

$$T_2 = \frac{F_3}{q^{\varphi_{B2}}-1} \left(\sum_{\kappa=0}^{\varphi_{B2}-1} \left(1 - \left\{ \frac{\kappa\overline{\psi}}{\varphi_{B2}} \right\} \right) q^{\kappa} - 1 \right),$$

and since $\{a\} = a - \lfloor a \rfloor$ for any $a \in \mathbf{R}$, we obtain that T_2 equals

$$\begin{aligned}
&\frac{F_3}{q^{\varphi_{B2}}-1} \left(\sum_{\kappa} q^{\kappa} - \frac{\overline{\psi}}{\varphi_{B2}} \sum_{\kappa} \kappa q^{\kappa} + \sum_{\kappa} \left\lfloor \frac{\kappa\overline{\psi}}{\varphi_{B2}} \right\rfloor q^{\kappa} - 1 \right) \\
&= \frac{F_3}{q^{\varphi_{B2}}-1} \left(\frac{q^{\varphi_{B2}}-1}{q-1} - \frac{\overline{\psi}}{\varphi_{B2}} \frac{q^{\varphi_{B2}}(\varphi_{B2}(q-1)-q)+q}{(q-1)^2} + \sum_{\kappa} \left\lfloor \frac{\kappa\overline{\psi}}{\varphi_{B2}} \right\rfloor q^{\kappa} - 1 \right) \\
&= \frac{F_3}{1-q} \left(\frac{\overline{\psi}}{1-q^{-\varphi_{B2}}} - \frac{\overline{\psi}}{\varphi_{B2}(1-q^{-1})} - 1 \right) + \frac{F_3}{q^{\varphi_{B2}}-1} \left(\sum_{\kappa} \left\lfloor \frac{\kappa\overline{\psi}}{\varphi_{B2}} \right\rfloor q^{\kappa} - 1 \right).
\end{aligned}$$

As we assume that $\varphi_{B2} > 1$, we have $\overline{\psi} \in \{1,\ldots,\varphi_{B2}-1\}$. Hence the finite sequence

$$\left(\left\lfloor \frac{\kappa\overline{\psi}}{\varphi_{B2}} \right\rfloor \right)_{\kappa=0}^{\varphi_{B2}-1} \tag{78}$$

of non-negative integers increases from 0 to $\overline{\psi}-1$ with steps of zero or one. If we denote

$$\kappa_\rho = \min \left\{ \kappa \in \mathbf{Z}_{\geqslant 0} \;\middle|\; \left\lfloor \frac{\kappa\overline{\psi}}{\varphi_{B2}} \right\rfloor = \rho \right\} = \left\lceil \frac{\rho\varphi_{B2}}{\overline{\psi}} \right\rceil; \qquad \rho = 0,\ldots,\overline{\psi};$$

then

$$0 = \kappa_0 < \kappa_1 < \cdots < \kappa_{\overline{\psi}-1} < \kappa_{\overline{\psi}} = \varphi_{B2},$$

and $\kappa_1,\ldots,\kappa_{\overline{\psi}-1}$ are the indices where a 'jump' in the sequence (78) takes place.

Let us express

$$\sum_{\kappa}\left\lfloor\frac{\kappa\overline{\psi}}{\varphi_{B2}}\right\rfloor q^{\kappa}$$

in terms of these numbers. We have

$$\begin{aligned}\left(\sum_{\kappa=0}^{\varphi_{B2}-1}\left\lfloor\frac{\kappa\overline{\psi}}{\varphi_{B2}}\right\rfloor q^{\kappa}\right)(1-q) &= \sum_{\kappa=1}^{\varphi_{B2}-1}\left(\left\lfloor\frac{\kappa\overline{\psi}}{\varphi_{B2}}\right\rfloor-\left\lfloor\frac{(\kappa-1)\overline{\psi}}{\varphi_{B2}}\right\rfloor\right)q^{\kappa}-(\overline{\psi}-1)q^{\varphi_{B2}}\\ &= \sum_{\rho=1}^{\overline{\psi}-1}q^{\kappa_\rho}-(\overline{\psi}-1)q^{\varphi_{B2}}\\ &= \sum_{\rho=1}^{\overline{\psi}}q^{\kappa_\rho}-\overline{\psi}q^{\varphi_{B2}},\end{aligned}$$

and therefore,

$$T_2=\frac{F_3}{1-q}\left(\frac{1}{q^{\varphi_{B2}}-1}\sum_{\rho=1}^{\overline{\psi}}q^{\kappa_\rho}-\frac{\overline{\psi}}{\varphi_{B2}(1-q^{-1})}-1\right)-\frac{F_3}{q^{\varphi_{B2}}-1}. \tag{79}$$

If we agree that an empty sum equals zero, then the above formula stays valid for $\varphi_{B2}=1$.

Case $n^*\nmid n$. With Equations (73) and (75) as a starting point, we now calculate $\Sigma_1^{(0)}$ in the complementary case. First of all, as $p^{(w\cdot v_0+\xi'_B(w\cdot v_1))/\mu_B}$ is now a μ_Bth root of unity different from one, one has that

$$\sum_{j=0}^{\mu_B-1}\left(p^{\frac{w\cdot v_0+\xi'_B(w\cdot v_1)}{\mu_B}}\right)^j=0,$$

and Expression (75) simplifies to

$$\begin{aligned}S(i,k)&=\left(p^{\frac{w\cdot v_0+\xi'_B(w\cdot v_1)}{\mu_B}}\right)^{-j_0}\sum_{j=0}^{\mu_B-1}j\left(p^{\frac{w\cdot v_0+\xi'_B(w\cdot v_1)}{\mu_B}}\right)^j\\ &=\frac{\mu_B}{p^{\frac{w\cdot v_0+\xi'_B(w\cdot v_1)}{\mu_B}}-1}\left(p^{\frac{w\cdot v_0+\xi'_B(w\cdot v_1)}{\mu_B}}\right)^{-j_0}.\end{aligned}$$

Next, let us recall from (72) and (59) that

$$j_0=\left\lfloor\frac{i\xi_A\mu_B\varphi_{AB}-k\Psi}{\mu_A\varphi_{AB}}\right\rfloor \qquad\text{and}\qquad \frac{\xi_A\mu_B+\xi_B\mu_A+\Psi}{\mu_A}\in\mu_B\mathbf{Z}.$$

We observe that

$$-j_0\equiv-\left\lfloor\frac{i(-\xi_B\mu_A-\Psi)\varphi_{AB}-k\Psi}{\mu_A\varphi_{AB}}\right\rfloor=i\xi_B-\left\lfloor\frac{-(i\varphi_{AB}+k)\Psi}{\mu_A\varphi_{AB}}\right\rfloor \mod \mu_B,$$

which gives rise to

$$S(i,k)=\frac{\mu_B}{p^{\frac{w\cdot v_0+\xi'_B(w\cdot v_1)}{\mu_B}}-1}\left(p^{\frac{\xi_B(w\cdot v_0)+w\cdot v_1}{\mu_B}}\right)^i\left(p^{\frac{w\cdot v_0+\xi'_B(w\cdot v_1)}{\mu_B}}\right)^{-\left\lfloor\frac{-(i\varphi_{AB}+k)\Psi}{\mu_A\varphi_{AB}}\right\rfloor}. \tag{80}$$

Finally, in Paragraph 4.13.3 we obtained

$$p^{\frac{\xi_A(w\cdot v_0)+w\cdot v_3}{\mu_A}}p^{\frac{\xi_B(w\cdot v_0)+w\cdot v_1}{\mu_B}}=p^{\varphi_{AB}(-s_0-1)}=q^{\varphi_{AB}};$$

using this identity, Formulas (73) and (80) for $\Sigma_1^{(0)}$ and $S(i,k)$ eventually yield

$$\Sigma_1^{(0)} = \frac{1}{p^{\frac{w\cdot v_0+\xi'_B(w\cdot v_1)}{\mu_B}} - 1} \sum_{i=0}^{\mu_A-1} \sum_{k=0}^{\varphi_{AB}-1} q^{i\varphi_{AB}+k}\Big(p^{\frac{w\cdot v_0+\xi'_B(w\cdot v_1)}{\mu_B}}\Big)^{-\left\lfloor \frac{-(i\varphi_{AB}+k)\Psi}{\mu_A\varphi_{AB}} \right\rfloor}.$$

Proceeding as in Case $n^* \mid n$, we write the double sum DS in the expression above as

$$\begin{aligned} DS &= \sum_{l=0}^{\mu_A\varphi_{AB}-1} q^l\Big(p^{\frac{w\cdot v_0+\xi'_B(w\cdot v_1)}{\mu_B}}\Big)^{-\left\lfloor \frac{-l\psi}{\varphi_{B2}} \right\rfloor} \\ &= \sum_{\iota=0}^{\mu_2-1} \sum_{\kappa=0}^{\varphi_{B2}-1} q^{\iota\varphi_{B2}+\kappa}\Big(p^{\frac{w\cdot v_0+\xi'_B(w\cdot v_1)}{\mu_B}}\Big)^{-\left\lfloor \frac{-(\iota\varphi_{B2}+\kappa)\psi}{\varphi_{B2}} \right\rfloor} \\ &= \sum_{\iota} \left[q^{\varphi_{B2}}\Big(p^{\frac{w\cdot v_0+\xi'_B(w\cdot v_1)}{\mu_B}}\Big)^{\psi}\right]^{\iota} \sum_{\kappa} q^{\kappa}\Big(p^{\frac{w\cdot v_0+\xi'_B(w\cdot v_1)}{\mu_B}}\Big)^{-\left\lfloor \frac{-\kappa\psi}{\varphi_{B2}} \right\rfloor}. \end{aligned}$$

By (62) and the fact that $\kappa\psi/\varphi_{B2} \notin \mathbf{Z}$ for $\kappa \in \{1,\ldots,\varphi_{B2}-1\}$,[27] we then have

$$DS = \sum_{\iota}\Big(p^{\frac{\xi_2(w\cdot v_1)+w\cdot v_3}{\mu_2}}\Big)^{\iota}\left[\sum_{\kappa} q^{\kappa}\Big(p^{\frac{w\cdot v_0+\xi'_B(w\cdot v_1)}{\mu_B}}\Big)^{\left\lfloor \frac{\kappa\psi}{\varphi_{B2}} \right\rfloor+1} - p^{\frac{w\cdot v_0+\xi'_B(w\cdot v_1)}{\mu_B}} + 1\right],$$

and hence we conclude

$$\begin{aligned} \Sigma_1^{(0)} &= \frac{p^{\frac{w\cdot v_0+\xi'_B(w\cdot v_1)}{\mu_B}} F_3}{\Big(p^{\frac{w\cdot v_0+\xi'_B(w\cdot v_1)}{\mu_B}} - 1\Big)\Big(p^{\frac{\xi_2(w\cdot v_1)+w\cdot v_3}{\mu_2}} - 1\Big)} \sum_{\kappa=0}^{\varphi_{B2}-1} q^{\kappa}\Big(p^{\frac{w\cdot v_0+\xi'_B(w\cdot v_1)}{\mu_B}}\Big)^{\left\lfloor \frac{\kappa\psi}{\varphi_{B2}} \right\rfloor} \\ &\quad - \frac{F_3}{p^{\frac{\xi_2(w\cdot v_1)+w\cdot v_3}{\mu_2}} - 1}. \end{aligned} \tag{81}$$

4.15.4. *A formula for* Σ'_1. Bringing together Equations (67, 69, 71, 77, and 79) for Σ'_1, $\Sigma_1^{(0)}$, $\Sigma_1^{(1)}$, $\Sigma_1^{(3)}$, and T_2, we find the following formula in case that $n^* \mid n$:

$$\begin{aligned} \frac{\Sigma'_1}{(\log p)F_3} &= \frac{m_0\Sigma_1^{(0)} + m_1\Sigma_1^{(1)} + m_3\Sigma_1^{(3)}}{F_3} \\ &= \frac{m_0}{1-q}\left(-\frac{\mu_B-1}{2} + \frac{1}{q^{\varphi_{B2}}-1}\sum_{\rho=1}^{\overline{\psi}} q^{\kappa_\rho} - \frac{\overline{\psi}}{\varphi_{B2}(1-q^{-1})} - 1\right) \\ &\quad - \frac{m_0}{q^{\varphi_{B2}}-1} + \frac{m_1}{1-q}\left(-\frac{\mu_B-1}{2} - \frac{1}{1-q^{-\varphi_{AB}}} + \frac{1}{\varphi_{AB}(1-q^{-1})}\right) \\ &\quad + \frac{m_3\mu_B}{1-q}\left(-\frac{F_3+1}{F_3} + \frac{1}{\mu_A\varphi_{AB}(1-q^{-1})}\right) \\ &= \frac{1}{1-q}\left(\frac{m_0}{q^{\varphi_{B2}}-1}\sum_{\rho=1}^{\overline{\psi}} q^{\kappa_\rho} - \frac{(m_0+m_1)(\mu_B-1)}{2} - \frac{m_1}{1-q^{-\varphi_{AB}}}\right. \\ &\quad \left. + \frac{m_1\mu_A + m_3\mu_B - m_0\overline{\psi}\mu_2}{\mu_A\varphi_{AB}(1-q^{-1})} - \frac{m_3\mu_B(F_3+1)}{F_3} - m_0\right) - \frac{m_0}{q^{\varphi_{B2}}-1}; \end{aligned} \tag{82}$$

[27] Recall that ψ and φ_{B2} are coprime.

if on the contrary $n^* \nmid n$, then Equations (67, 70, and 81) yield

$$
\begin{aligned}
\frac{\Sigma_1'}{(\log p)F_3} &= \frac{m_0\Sigma_1^{(0)} + m_1\Sigma_1^{(1)} + m_3\Sigma_1^{(3)}}{F_3} \\
&= \frac{m_0 p^{\frac{w\cdot v_0+\xi_B'(w\cdot v_1)}{\mu_B}}}{\left(p^{\frac{w\cdot v_0+\xi_B'(w\cdot v_1)}{\mu_B}}-1\right)\left(p^{\frac{\xi_2(w\cdot v_1)+w\cdot v_3}{\mu_2}}-1\right)} \sum_{\kappa=0}^{\varphi_{B2}-1} q^{\kappa}\left(p^{\frac{w\cdot v_0+\xi_B'(w\cdot v_1)}{\mu_B}}\right)^{\left\lfloor \frac{\kappa\psi}{\varphi_{B2}}\right\rfloor} \\
&\quad - \frac{m_0}{p^{\frac{\xi_2(w\cdot v_1)+w\cdot v_3}{\mu_2}}-1} + \frac{m_1}{q-1}\frac{q^{\varphi_{AB}}-1}{\left(p^{\frac{\xi_A(w\cdot v_0)+w\cdot v_3}{\mu_A}}-1\right)\left(p^{\frac{\xi_B(w\cdot v_0)+w\cdot v_1}{\mu_B}}-1\right)}.
\end{aligned}
\tag{83}
$$

4.16. **An easier formula for the residue R_1.**

4.16.1. *Case $n^* \mid n$.* If we fill in Formula (82) in Equation (65) for R_1', the latter simplifies to

$$
\frac{1}{1-q}\left(\frac{m_0}{q^{\varphi_{B2}}-1}\sum_{\rho=1}^{\overline{\psi}} q^{\kappa_\rho} + \frac{m_0}{q^{\varphi_{BC}}-1} + \frac{m_1\mu_A + m_3\mu_B + \mu_1 - m_0\overline{\psi}\mu_2}{\mu_A\varphi_{AB}(1-q^{-1})}\right) + \frac{m_0\Sigma_3}{F_2F_3}.
$$

There is a very convenient interpretation of $m_1\mu_A + m_3\mu_B + \mu_1$ appearing in the equation above. Recall from (44) that

$$
\Psi v_0 - \mu_A v_1 - \mu_B v_3 = (0, 0, \mu_1).
$$

Making the dot product with $D = (x_D, y_D, 1)$ on both sides yields

$$
m_0\Psi - m_1\mu_A - m_3\mu_B = \Psi(D\cdot v_0) - \mu_A(D\cdot v_1) - \mu_B(D\cdot v_3) = D\cdot(0,0,\mu_1) = \mu_1,
$$

and hence

$$
m_1\mu_A + m_3\mu_B + \mu_1 = m_0\Psi = m_0\psi\mu_2.
$$

It follows that

$$
\frac{m_1\mu_A + m_3\mu_B + \mu_1 - m_0\overline{\psi}\mu_2}{\mu_A\varphi_{AB}(1-q^{-1})} = \frac{m_0(\psi-\overline{\psi})}{\varphi_{B2}(1-q^{-1})} = \frac{m_0 t}{1-q^{-1}},
$$

with

$$
t = \frac{\psi-\overline{\psi}}{\varphi_{B2}} = \frac{\psi - \{\psi\}_{\varphi_{B2}}}{\varphi_{B2}} = \left\lfloor \frac{\psi}{\varphi_{B2}} \right\rfloor
$$

the quotient of Euclidean division of ψ by φ_{B2}. Note that if $\varphi_{B2} = 1$, then $t = \psi$.

If we now put $R_1'' = R_1'/m_0$, it remains to prove that

$$
R_1'' = \frac{\Sigma_3}{F_2F_3} + \frac{1}{1-q}\left(\frac{1}{q^{\varphi_{B2}}-1}\sum_{\rho=1}^{\overline{\psi}} q^{\kappa_\rho} + \frac{1}{q^{\varphi_{BC}}-1} + \frac{t}{1-q^{-1}}\right)
\tag{84}
$$

vanishes.

4.16.2. *Case $n^* \nmid n$.* In this case, according to (66) and (83), it now comes to proving that

$$R_1'' = \frac{R_1'}{m_0} = \frac{\Sigma_3}{F_2F_3} - \frac{q^{\varphi_{BC}} - 1}{(q-1)\left(p^{\frac{w\cdot v_0 + \xi_B'(w\cdot v_1)}{\mu_B}} - 1\right)\left(p^{\frac{\xi_C(w\cdot v_1) + w\cdot v_2}{\mu_C}} - 1\right)} + \frac{p^{\frac{w\cdot v_0+\xi_B'(w\cdot v_1)}{\mu_B}}}{\left(p^{\frac{w\cdot v_0+\xi_B'(w\cdot v_1)}{\mu_B}} - 1\right)\left(p^{\frac{\xi_2(w\cdot v_1)+w\cdot v_3}{\mu_2}} - 1\right)} \sum_{\kappa=0}^{\varphi_{B2}-1} q^\kappa \left(p^{\frac{w\cdot v_0+\xi_B'(w\cdot v_1)}{\mu_B}}\right)^{\left\lfloor \frac{\kappa\psi}{\varphi_{B2}} \right\rfloor} \tag{85}$$

equals zero.

4.17. **Investigation of Σ_3.** First we try to find a useful formula for μ_3.

4.17.1. *Multiplicity μ_3 of δ_3.* From our study in Section 1, we remember that

$$\mu_C\mu_2 = \#H(v_1,v_2)\#H(v_1,v_3) \mid \mu_3 = \#H(v_1,v_2,v_3).$$

We look for more information on the factor $\varphi_{C2} = \mu_3/\mu_C\mu_2$. Let us proceed in the same way as when interpreting $\mu_C/\mu_A\mu_B = \varphi_{AB}$ in Case I (cfr. Subsection 2.12).

One has

$$\begin{aligned}
\mu_3 &= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \\
&= a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \\
&= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} (a_3\alpha_C + b_3\beta_C + c_3) \\
&= -\mu_C \left(v_3 \cdot \overrightarrow{CD}\right),
\end{aligned}$$

and since v_3 and $\overrightarrow{AD}$ are perpendicular, we can continue:

$$\mu_3 = \mu_C\left(v_3 \cdot \overrightarrow{AB} + v_3 \cdot \overrightarrow{BC}\right).$$

The fact that $\overrightarrow{AB} \perp v_0$ and $\overrightarrow{BC} \perp v_1$ implies that

$$\overrightarrow{AB} = \varphi_{AB}(-b_0, a_0, 0) \qquad \text{and} \qquad \overrightarrow{BC} = \varphi_{BC}(-b_1, a_1, 0).$$

Hence

$$\begin{aligned}
\mu_3 &= \mu_C\big(\varphi_{AB}(a_0b_3 - a_3b_0) + \varphi_{BC}(a_1b_3 - a_3b_1)\big) \\
&= \mu_C(\mu_A\varphi_{AB} + \varphi_{BC}\Psi) \\
&= \mu_C\mu_2\varphi_{C2},
\end{aligned}$$

with

$$\varphi_{C2} = \varphi_{B2} + \varphi_{BC}\psi. \tag{86}$$

Note that (86) and the coprimality of ψ and φ_{B2} imply $\psi \in \{0, \ldots, \varphi_{C2} - 1\}$ and $\gcd(\psi, \varphi_{C2}) = 1$.

Next, we try to list all the $\mu_3 = \mu_C\mu_2\varphi_{C2}$ points of H_3.

4.17.2. *Points of* $H_3 = H(v_1, v_2, v_3)$. We proceed in the same way as in Case I for the points of H_C. As we know, the points of $H_C = H(v_1, v_2)$ and $H_2 = H(v_1, v_3)$ can be presented as

$$h(i,0,0) = \left\{ \frac{i\xi_C}{\mu_C} \right\} v_1 + \frac{i}{\mu_C} v_2; \qquad i = 0, \ldots, \mu_C - 1;$$

and

$$h(0,j,0) = \left\{ \frac{j\xi_2}{\mu_2} \right\} v_1 + \frac{j}{\mu_2} v_3; \qquad j = 0, \ldots, \mu_2 - 1;$$

respectively.

To generate a complete list of points of H_3, it is now sufficient to find a set of representatives for the φ_{C2} cosets of the subgroup $H_C + H_2$ of H_3. Recall that the cosets of $H_C + H_2$ can be described as[28]

$$\mathcal{C}_k = \left\{ h \in H_3 \;\middle|\; \{h_3\}_{\frac{1}{\mu_2}} = \frac{k\mu_C}{\mu_3} = \frac{k}{\mu_2 \varphi_{C2}} \right\}; \qquad k = 0, \ldots, \varphi_{C2} - 1.$$

We will follow the approach of Section 1 and select for each coset $\mathcal{C}_k$, as a representative, the unique element $h(0,0,k) \in \mathcal{C}_k$ with v_3-coordinate $h_3(0,0,k) = k/\mu_2\varphi_{C2}$ and v_2-coordinate $h_2(0,0,k) < 1/\mu_C$. We find $h(0,0,1)$ as follows.

Recall from (46) that

$$\Theta v_1 - \Psi v_2 - \mu_C v_3 = (0,0,\mu_3),$$

with $\Theta = \left| \begin{smallmatrix} a_2 & b_2 \\ a_3 & b_3 \end{smallmatrix} \right| > 0$. It follows that

$$h(0,0,1) = \left\{ \frac{-\Theta}{\mu_3} \right\} v_1 + \frac{\psi}{\mu_C \varphi_{C2}} v_2 + \frac{1}{\mu_2 \varphi_{C2}} v_3 = \{(0,0,-1)\} \in \mathcal{C}_1$$

is the representative for $\mathcal{C}_1$ we are looking for. Indeed, it follows from Equation (86) that $h_2(0,0,1) = \psi/\mu_C\varphi_{C2}$ is not only reduced modulo 1, it is also already reduced modulo $1/\mu_C$.

It is now natural to find all representatives $h(0,0,k)$ by considering the φ_{C2} multiples $\{kh(0,0,1)\}$; $k = 0, \ldots, \varphi_{C2} - 1$; of $h(0,0,1)$ in the group H_3, and adding to each multiple $\{kh(0,0,1)\}$ the unique element of H_C such that the v_2-coordinate $h_2(0,0,k)$ of the sum $h(0,0,k)$ is reduced modulo $1/\mu_C$:

$$h(0,0,k) = \left\{ \frac{-k\Theta - \lfloor k\psi/\varphi_{C2} \rfloor \xi_C \mu_2 \varphi_{C2}}{\mu_3} \right\} v_1 + \frac{\{k\psi\}_{\varphi_{C2}}}{\mu_C \varphi_{C2}} v_2 + \frac{k}{\mu_2 \varphi_{C2}} v_3 \in \mathcal{C}_k;$$
$$k = 0, \ldots, \varphi_{C2} - 1.$$

Note that since ψ and φ_{C2} are coprime, $\{k\psi\}_{\varphi_{C2}}$ runs, as expected, through the numbers $0, \ldots, \varphi_{C2} - 1$ when k does so.

[28] Here h_3 denotes the v_3-coordinate of h. We can as well, and completely similarly, describe these cosets in terms of the v_2-coordinate, but the choice for h_3 is more convenient in this case.

All this leads to the following member list of H_3:

$$\begin{aligned} h(i,j,k) &= \{h(i,0,0)+h(0,j,0)+h(0,0,k)\} \\ &= \left\{\frac{(i-\lfloor k\psi/\varphi_{C2}\rfloor)\xi_C\mu_2\varphi_{C2}+j\xi_2\mu_C\varphi_{C2}-k\Theta}{\mu_3}\right\}v_1 \\ &\qquad\qquad + \frac{i\varphi_{C2}+\{k\psi\}_{\varphi_{C2}}}{\mu_C\varphi_{C2}}v_2+\frac{j\varphi_{C2}+k}{\mu_2\varphi_{C2}}v_3; \\ &\qquad i=0,\dots,\mu_C-1;\quad j=0,\dots,\mu_2-1;\quad k=0,\dots,\varphi_{C2}-1. \end{aligned}$$

Finally, we will try to calculate Σ_3 based on the above description of H_3's points.

4.17.3. *Calculation of* Σ_3. Writing h as $h=h_1v_1+h_2v_2+h_3v_3$ for $h\in H_3=H(v_1,v_2,v_3)$ and noting that $p^{w\cdot v_1}=1$, we find

$$\begin{aligned} \Sigma_3 &= \sum_{h\in H_3} p^{w\cdot h} \\ &= \sum_{i=0}^{\mu_C-1}\sum_{j=0}^{\mu_2-1}\sum_{k=0}^{\varphi_{C2}-1} \\ &\qquad p^{\frac{(i-\lfloor k\psi/\varphi_{C2}\rfloor)\xi_C\mu_2\varphi_{C2}+j\xi_2\mu_C\varphi_{C2}-k\Theta}{\mu_3}(w\cdot v_1)+\frac{i\varphi_{C2}+\{k\psi\}_{\varphi_{C2}}}{\mu_C\varphi_{C2}}(w\cdot v_2)+\frac{j\varphi_{C2}+k}{\mu_2\varphi_{C2}}(w\cdot v_3)} \\ &= \sum_i\Big(p^{\frac{\xi_C(w\cdot v_1)+w\cdot v_2}{\mu_C}}\Big)^i\sum_j\Big(p^{\frac{\xi_2(w\cdot v_1)+w\cdot v_3}{\mu_2}}\Big)^j \\ &\qquad\qquad \sum_k\Big(p^{\frac{-\Theta(w\cdot v_1)+\Psi(w\cdot v_2)+\mu_C(w\cdot v_3)}{\mu_3}}\Big)^k\Big(p^{\frac{\xi_C(w\cdot v_1)+w\cdot v_2}{\mu_C}}\Big)^{-\left\lfloor\frac{k\psi}{\varphi_{C2}}\right\rfloor} \end{aligned}$$

$$= \frac{F_2F_3}{\Big(p^{\frac{\xi_C(w\cdot v_1)+w\cdot v_2}{\mu_C}}-1\Big)\Big(p^{\frac{\xi_2(w\cdot v_1)+w\cdot v_3}{\mu_2}}-1\Big)}\sum_{k=0}^{\varphi_{C2}-1}q^k\Big(p^{\frac{\xi_C(w\cdot v_1)+w\cdot v_2}{\mu_C}}\Big)^{-\left\lfloor\frac{k\psi}{\varphi_{C2}}\right\rfloor}, \tag{87}$$

where in the last step we used Identity (49).

In the special case that $n^*\mid n$, based on (61–63), we obtain the slightly simpler formula

$$\Sigma_3=\frac{F_2F_3}{(q^{\varphi_{BC}}-1)(q^{\varphi_{B2}}-1)}\sum_{k=0}^{\varphi_{C2}-1}q^{k-\varphi_{BC}\left\lfloor\frac{k\psi}{\varphi_{C2}}\right\rfloor}. \tag{88}$$

4.18. **Proof that the residue R_1 equals zero.**

4.18.1. *Case* $n^*\mid n$. According to Formula (84) for R_1'' and Formula (88) for Σ_3, it now suffices to prove that

$$\sum_{k=0}^{\varphi_{C2}-1}q^{k-\varphi_{BC}\left\lfloor\frac{k\psi}{\varphi_{C2}}\right\rfloor}=\frac{q^{\varphi_{BC}}-1}{q-1}\sum_{\rho=1}^{\overline{\psi}}q^{\kappa_\rho}+\frac{q^{\varphi_{B2}}-1}{q-1}\left(tq\frac{q^{\varphi_{BC}}-1}{q-1}+1\right). \tag{89}$$

Let us do this now.

First of all, if $\varphi_{B2} = 1$, then by (86) we have $\varphi_{C2} = \varphi_{BC}\psi + 1$, and

$$\begin{aligned}
\sum_{k=0}^{\varphi_{C2}-1} q^{k-\varphi_{BC}\left\lfloor \frac{k\psi}{\varphi_{C2}} \right\rfloor} &= 1 + \sum_{k=1}^{\psi\varphi_{BC}} q^{k-\varphi_{BC}\left\lfloor \frac{k\psi}{\varphi_{C2}} \right\rfloor} \\
&= 1 + \sum_{r=0}^{\psi-1} \sum_{l=1}^{\varphi_{BC}} q^{(r\varphi_{BC}+l)-\varphi_{BC}\left\lfloor \frac{(r\varphi_{BC}+l)\psi}{\varphi_{BC}\psi+1} \right\rfloor} \\
&= 1 + \sum_{r} \sum_{l} q^{l} \\
&= \psi q \frac{q^{\varphi_{BC}} - 1}{q-1} + 1,
\end{aligned}$$

which agrees with (89) for $\varphi_{B2} = 1$.

In what follows, we shall assume that $\varphi_{B2} > 1$ and thus that $\overline{\psi} > 0$. Since $\psi \in \{1, \ldots, \varphi_{C2} - 1\}$, the finite sequence

$$\left(\left\lfloor \frac{k\psi}{\varphi_{C2}} \right\rfloor \right)_{k=0}^{\varphi_{C2}-1} \tag{90}$$

of non-negative integers increases from 0 to $\psi - 1$ with steps of zero or one. Let us denote

$$k_r = \min\left\{ k \in \mathbf{Z}_{\geqslant 0} \;\middle|\; \left\lfloor \frac{k\psi}{\varphi_{C2}} \right\rfloor = r \right\} = \left\lceil \frac{r\varphi_{C2}}{\psi} \right\rceil; \qquad r = 0, \ldots, \psi.$$

Then

$$0 = k_0 < k_1 < \cdots < k_{\psi-1} < k_\psi = \varphi_{C2},$$

and obviously,

$$\sum_{k=0}^{\varphi_{C2}-1} q^{k-\varphi_{BC}\left\lfloor \frac{k\psi}{\varphi_{C2}} \right\rfloor} = \sum_{r=0}^{\psi-1} \sum_{k=k_r}^{k_{r+1}-1} q^{k-\varphi_{BC} r}.$$

We recall that

$$\varphi_{C2} = \varphi_{B2} + \varphi_{BC}\psi \qquad \text{and} \qquad \psi = t\varphi_{B2} + \overline{\psi}, \tag{91}$$

with $t \in \mathbf{Z}_{\geqslant 0}$ and $\overline{\psi} = \{\psi\}_{\varphi_{B2}} \in \{1, \ldots, \varphi_{B2} - 1\}$. Remember also that for $\rho \in \{0, \ldots, \overline{\psi}\}$, the number κ_ρ denotes the smallest integer satisfying $\kappa_\rho \overline{\psi} \geqslant \rho\varphi_{B2}$.

Let us first verify (89) for $t = 0$. In this case we have that $\psi = \overline{\psi}$, and hence

$$k_\rho = \left\lceil \frac{\rho\varphi_{C2}}{\overline{\psi}} \right\rceil = \left\lceil \frac{\rho(\varphi_{B2} + \varphi_{BC}\overline{\psi})}{\overline{\psi}} \right\rceil = \varphi_{BC}\rho + \left\lceil \frac{\rho\varphi_{B2}}{\overline{\psi}} \right\rceil = \varphi_{BC}\rho + \kappa_\rho,$$

for all $\rho \in \{0, \ldots, \overline{\psi}\}$. It follows that

$$\begin{aligned}
\sum_{k=0}^{\varphi_{C2}-1} q^{k-\varphi_{BC}\left\lfloor \frac{k\psi}{\varphi_{C2}} \right\rfloor} &= \sum_{\rho=0}^{\overline{\psi}-1} \sum_{k=k_\rho}^{k_{\rho+1}-1} q^{k-\varphi_{BC}\rho} \\
&= \sum_{\rho=0}^{\overline{\psi}-1} \sum_{k=\varphi_{BC}\rho+\kappa_\rho}^{\varphi_{BC}(\rho+1)+\kappa_{\rho+1}-1} q^{k-\varphi_{BC}\rho} \\
&= \sum_{\rho=0}^{\overline{\psi}-1} \sum_{\kappa=\kappa_\rho}^{\varphi_{BC}+\kappa_{\rho+1}-1} q^{\kappa} \\
&= \sum_{\rho=0}^{\overline{\psi}-1} \left(\sum_{\kappa=\kappa_\rho}^{\kappa_{\rho+1}-1} q^{\kappa} + q^{\kappa_{\rho+1}} \sum_{l=0}^{\varphi_{BC}-1} q^{l} \right) \\
&= \sum_{\kappa=0}^{\varphi_{B2}-1} q^{\kappa} + \frac{q^{\varphi_{BC}}-1}{q-1} \sum_{\rho=0}^{\overline{\psi}-1} q^{\kappa_{\rho+1}} \\
&= \frac{q^{\varphi_{BC}}-1}{q-1} \sum_{\rho=1}^{\overline{\psi}} q^{\kappa_\rho} + \frac{q^{\varphi_{B2}}-1}{q-1},
\end{aligned}$$

which agrees with (89) for $t = 0$.

Let us from now on assume that $t > 0$. In the lemma below we express k_r and $k_{r+1} - 1$ explicitly as a function of r after writing r in a special form, but first we introduce the following notation.

Notation 4.2 (Iverson's convention). Cfr. [28]. For any proposition P we shall denote by

$$[P] = \begin{cases} 1, & \text{if } P \text{ is true;} \\ 0, & \text{otherwise;} \end{cases}$$

the truth value of P.

Lemma 4.3. *Assume that $t > 0$. Then the map*

$$\begin{aligned}
&\big\{(\rho, \kappa, \lambda) \in \mathbf{Z}^3_{\geqslant 0} \mid \\
&\quad 0 \leqslant \rho \leqslant \overline{\psi} - 1,\ \ \kappa_\rho \leqslant \kappa \leqslant \kappa_{\rho+1} - 1,\ \ 0 \leqslant \lambda \leqslant t - [\kappa < \kappa_{\rho+1} - 1]\big\} \\
&\qquad\qquad \to \{0, \ldots, \psi - 1\} : (\rho, \kappa, \lambda) \mapsto r = \kappa t + \rho + \lambda
\end{aligned} \tag{92}$$

is bijective, and for $r = \kappa t + \rho + \lambda \in \{0, \ldots, \psi - 1\}$ written in this way, we have that

$$\begin{aligned}
k_r &= \varphi_{BC} r + \kappa + [\lambda > 0] \qquad \textit{and} \\
k_{r+1} - 1 &= \varphi_{BC}(r+1) + \kappa.
\end{aligned}$$

We will prove this lemma shortly. If we accept it now, we obtain[29]

$$\begin{aligned}
\sum_{k=0}^{\varphi_{C2}-1} q^{k-\varphi_{BC}\left\lfloor \frac{k\psi}{\varphi_{C2}}\right\rfloor} &= \sum_{r=0}^{\psi-1}\sum_{k=k_r}^{k_{r+1}-1} q^{k-\varphi_{BC}r} \\
&= \sum_{\rho=0}^{\overline{\psi}-1}\sum_{\kappa=\kappa_\rho}^{\kappa_{\rho+1}-1}\sum_{\lambda=0}^{t-[\kappa<\kappa_{\rho+1}-1]}\sum_{k=\varphi_{BC}r+\kappa+[\lambda>0]}^{\varphi_{BC}(r+1)+\kappa} q^{k-\varphi_{BC}r} \\
&= \sum_{\rho=0}^{\overline{\psi}-1}\sum_{\kappa=\kappa_\rho}^{\kappa_{\rho+1}-1} q^{\kappa}\sum_{\lambda=0}^{t-[\kappa<\kappa_{\rho+1}-1]}\sum_{l=[\lambda>0]}^{\varphi_{BC}} q^{l} \\
&= \sum_{\rho=0}^{\overline{\psi}-1}\sum_{\kappa=\kappa_\rho}^{\kappa_{\rho+1}-1} q^{\kappa}\sum_{\lambda=0}^{t-[\kappa<\kappa_{\rho+1}-1]}\left(q\frac{q^{\varphi_{BC}}-1}{q-1}+[\lambda=0]\right) \\
&= \sum_{\rho=0}^{\overline{\psi}-1}\sum_{\kappa=\kappa_\rho}^{\kappa_{\rho+1}-1} q^{\kappa}\left(\big(t+[\kappa=\kappa_{\rho+1}-1]\big)q\frac{q^{\varphi_{BC}}-1}{q-1}+1\right) \\
&= \sum_{\rho=0}^{\overline{\psi}-1}\left[\left(tq\frac{q^{\varphi_{BC}}-1}{q-1}+1\right)\sum_{\kappa=\kappa_\rho}^{\kappa_{\rho+1}-1} q^{\kappa}+q^{\kappa_{\rho+1}}\frac{q^{\varphi_{BC}}-1}{q-1}\right] \\
&= \left(tq\frac{q^{\varphi_{BC}}-1}{q-1}+1\right)\sum_{\kappa=0}^{\varphi_{B2}-1} q^{\kappa}+\frac{q^{\varphi_{BC}}-1}{q-1}\sum_{\rho=0}^{\overline{\psi}-1} q^{\kappa_{\rho+1}} \\
&= \frac{q^{\varphi_{BC}}-1}{q-1}\sum_{\rho=1}^{\overline{\psi}} q^{\kappa_\rho}+\frac{q^{\varphi_{B2}}-1}{q-1}\left(tq\frac{q^{\varphi_{BC}}-1}{q-1}+1\right),
\end{aligned}$$

which agrees with (89).

We conclude the proof of $R_1 = 0$ in Case $n^* \mid n$ by verifying Lemma 4.3. Since for all (ρ, κ, λ) in the domain, we have that

$$0 \leqslant \kappa t + \rho + \lambda \leqslant (\kappa_{\overline{\psi}} - 1)t + (\overline{\psi} - 1) + t = t\varphi_{B2} + \overline{\psi} - 1 = \psi - 1,$$

the map (92) is well-defined.

We check that the map is onto. Let $r \in \{0, \ldots, \psi - 1\}$. Because the finite sequence

$$\big(\kappa_\rho t + \rho\big)_{\rho=0}^{\overline{\psi}}$$

of non-negative integers strictly increases from $\kappa_0 t + 0 = 0$ to $\kappa_{\overline{\psi}} t + \overline{\psi} = \psi$, there exists a (unique) $\rho \in \{0, \ldots, \overline{\psi} - 1\}$ such that

$$\kappa_\rho t + \rho \leqslant r < \kappa_{\rho+1} t + (\rho + 1).$$

If $r = \kappa_{\rho+1} t + \rho$, we can write r as $r = (\kappa_{\rho+1} - 1)t + \rho + t$, and r is the image of $(\rho, \kappa_{\rho+1} - 1, t)$ under the map (92). Otherwise we have that

$$\kappa_\rho t \leqslant r - \rho < \kappa_{\rho+1} t,$$

[29] Note that $r = \kappa t + \rho + \lambda$ in the second line.

and we can write $r - \rho$ (in a unique way) as $r - \rho = \kappa t + \lambda$ with $\kappa, \lambda \in \mathbf{Z}$; $\kappa_\rho \leqslant \kappa \leqslant \kappa_{\rho+1} - 1$; and $0 \leqslant \lambda \leqslant t - 1$. In this case $r = \kappa t + \rho + \lambda$ is the image of (ρ, κ, λ) under the map (92). This proves surjectivity.

The uniqueness of the representation $r = \kappa t + \rho + \lambda$ can either be checked directly, or by verifying that the cardinality of the domain,

$$\begin{aligned}\sum_{\rho=0}^{\overline{\psi}-1} \sum_{\kappa=\kappa_\rho}^{\kappa_{\rho+1}-1} \sum_{\lambda=0}^{t-[\kappa<\kappa_{\rho+1}-1]} 1 &= \sum_{\rho=0}^{\overline{\psi}-1} \sum_{\kappa=\kappa_\rho}^{\kappa_{\rho+1}-1} \big(t + [\kappa = \kappa_{\rho+1} - 1]\big) \\ &= \sum_{\kappa=0}^{\varphi_{B2}-1} t + \sum_{\rho=0}^{\overline{\psi}-1} 1 \\ &= t\varphi_{B2} + \overline{\psi} \\ &= \psi,\end{aligned}$$

indeed equals the cardinality of the codomain $\{0, \dots, \psi - 1\}$.

Let $r = \kappa t + \rho + \lambda \in \{0, \dots, \psi - 1\}$, written in the appropriate way. We prove the expression for k_r stated in the lemma. On the one hand, because $\kappa \geqslant \kappa_\rho$, it holds that $\kappa\overline{\psi} \geqslant \kappa_\rho\overline{\psi} \geqslant \rho\varphi_{B2}$, and since $\lambda \leqslant t$, we have

$$(\rho + \lambda)\varphi_{B2} \leqslant \kappa\overline{\psi} + [\lambda > 0](t\varphi_{B2} + \overline{\psi}).$$

On the other hand, since we assume $t > 0$, it follows from $\kappa \leqslant \kappa_{\rho+1} - 1$ that

$$\kappa\overline{\psi} \leqslant (\kappa_{\rho+1} - 1)\overline{\psi} < (\rho + 1)\varphi_{B2} \leqslant (\rho + \lambda)\varphi_{B2} + [\lambda = 0](t\varphi_{B2} + \overline{\psi}).$$

Hence

$$\kappa\overline{\psi} - [\lambda = 0](t\varphi_{B2} + \overline{\psi}) < (\rho + \lambda)\varphi_{B2} \leqslant \kappa\overline{\psi} + [\lambda > 0](t\varphi_{B2} + \overline{\psi}).$$

Adding $\kappa t\varphi_{B2}$ in all sides of the equation, we get

$$(\kappa - [\lambda = 0])(t\varphi_{B2} + \overline{\psi}) < (\kappa t + \rho + \lambda)\varphi_{B2} \leqslant (\kappa + [\lambda > 0])(t\varphi_{B2} + \overline{\psi}).$$

If we apply (91) and the representation of r, we obtain

$$(\kappa - [\lambda = 0])\psi < r\varphi_{B2} \leqslant (\kappa + [\lambda > 0])\psi,$$

and after adding $\varphi_{BC}r\psi$, we have

$$(\varphi_{BC}r + \kappa - [\lambda = 0])\psi < r(\varphi_{B2} + \varphi_{BC}\psi) \leqslant (\varphi_{BC}r + \kappa + [\lambda > 0])\psi.$$

Using Formula (91) for φ_{C2}, we eventually obtain

$$(\varphi_{BC}r + \kappa + [\lambda > 0] - 1)\psi < r\varphi_{C2} \leqslant (\varphi_{BC}r + \kappa + [\lambda > 0])\psi,$$

which proves that

$$k_r = \varphi_{BC}r + \kappa + [\lambda > 0]. \tag{93}$$

Finally, let us verify the expression for $k_{r+1} - 1$. If $r = \psi - 1$, then

$$r = (\kappa_{\overline{\psi}} - 1)t + (\overline{\psi} - 1) + t \qquad \text{and}$$

$$\begin{aligned}k_{r+1} - 1 = k_\psi - 1 = \varphi_{C2} - 1 &= \varphi_{B2} + \varphi_{BC}\psi - 1 \\ &= \varphi_{BC}\psi + (\kappa_{\overline{\psi}} - 1) = \varphi_{BC}(r + 1) + \kappa.\end{aligned}$$

Otherwise $r+1 \leqslant \psi - 1$ and we can use (93) to find k_{r+1}. First suppose that $\lambda < t - [\kappa < \kappa_{\rho+1} - 1]$. Then we have

$$\begin{gathered} r+1 = \kappa t + \rho + (\lambda+1) \qquad \text{and} \\ k_{r+1} - 1 = \varphi_{BC}(r+1) + \kappa + [\lambda+1 > 0] - 1 = \varphi_{BC}(r+1) + \kappa. \end{gathered}$$

If on the contrary $\lambda = t - [\kappa < \kappa_{\rho+1} - 1]$, we have

$$\begin{gathered} r+1 = (\kappa+1)t + (\rho + [\kappa = \kappa_{\rho+1} - 1]) + 0 \qquad \text{and again} \\ k_{r+1} - 1 = \varphi_{BC}(r+1) + (\kappa+1) + [0 > 0] - 1 = \varphi_{BC}(r+1) + \kappa. \end{gathered}$$

This ends the proof of the lemma and concludes Case $n^* \mid n$.

4.18.2. *Case* $n^* \nmid n$. By Equations (85) and (87) for R_1'' and Σ_3, proving $R_1'' = 0$ boils down to verifying that

$$\begin{aligned} &\left(p^{\frac{w\cdot v_0 + \xi_B'(w\cdot v_1)}{\mu_B}} - 1\right) \sum_{k=0}^{\varphi_{C2}-1} q^k \left(p^{\frac{\xi_C(w\cdot v_1) + w\cdot v_2}{\mu_C}}\right)^{-\left\lfloor \frac{k\psi}{\varphi_{C2}} \right\rfloor} \\ &\qquad + p^{\frac{w\cdot v_0 + \xi_B'(w\cdot v_1)}{\mu_B}} \left(p^{\frac{\xi_C(w\cdot v_1) + w\cdot v_2}{\mu_C}} - 1\right) \sum_{\kappa=0}^{\varphi_{B2}-1} q^\kappa \left(p^{\frac{w\cdot v_0 + \xi_B'(w\cdot v_1)}{\mu_B}}\right)^{\left\lfloor \frac{\kappa\psi}{\varphi_{B2}} \right\rfloor} \\ &\qquad\qquad = \left(p^{\frac{\xi_2(w\cdot v_1) + w\cdot v_3}{\mu_2}} - 1\right) \frac{q^{\varphi_{BC}} - 1}{q-1}. \end{aligned}$$

Expressing everything in terms of

$$q \qquad \text{and} \qquad \beta = p^{\frac{w\cdot v_0 + \xi_B'(w\cdot v_1)}{\mu_B}}$$

by means of Identities (61) and (62), the above statement is equivalent to

(94)
$$\begin{aligned} (q-1)(\beta-1) \sum_{k=0}^{\varphi_{C2}-1} q^{k - \varphi_{BC}\left\lfloor \frac{k\psi}{\varphi_{C2}} \right\rfloor} \beta^{\left\lfloor \frac{k\psi}{\varphi_{C2}} \right\rfloor} + (q-1)(q^{\varphi_{BC}} - \beta) \sum_{\kappa=0}^{\varphi_{B2}-1} q^\kappa \beta^{\left\lfloor \frac{\kappa\psi}{\varphi_{B2}} \right\rfloor} \\ = (q^{\varphi_{BC}} - 1)(q^{\varphi_{B2}} \beta^\psi - 1). \end{aligned}$$

This equality in fact turns out to be a polynomial identity in the variables q and β, as we will show now.

Both sequences, $\left(\lfloor k\psi/\varphi_{C2} \rfloor\right)_{k=0}^{\varphi_{C2}}$ and $\left(\lfloor \kappa\psi/\varphi_{B2} \rfloor\right)_{\kappa=0}^{\varphi_{B2}}$, increase from 0 to ψ. As $\psi < \varphi_{C2}$ and ψ may be strictly greater than φ_{B2}, the first sequence adopts all values in $\{0, \ldots, \psi\}$, but the second one may not. We put

$$\begin{aligned} k_r &= \min\left\{ k \in \mathbf{Z}_{\geqslant 0} \;\middle|\; \left\lfloor \frac{k\psi}{\varphi_{C2}} \right\rfloor = r \right\} = \left\lceil \frac{r\varphi_{C2}}{\psi} \right\rceil \qquad \text{and} \\ \kappa_r &= \min\left\{ \kappa \in \mathbf{Z}_{\geqslant 0} \;\middle|\; \left\lfloor \frac{\kappa\psi}{\varphi_{B2}} \right\rfloor \geqslant r \right\} = \left\lceil \frac{r\varphi_{B2}}{\psi} \right\rceil; \qquad r = 0, \ldots, \psi. \end{aligned}$$

The numbers k_r are the same as in Case $n^* \mid n$, while the κ_r are defined differently; note that the sequence $(\kappa_r)_r$ is still increasing, but no longer necessarily strictly. We have

$$\begin{gathered} 0 = k_0 < k_1 < \cdots < k_{\psi-1} < k_\psi = \varphi_{C2} \qquad \text{and} \\ 0 = \kappa_0 < \kappa_1 \leqslant \cdots \leqslant \kappa_{\psi-1} \leqslant \kappa_\psi = \varphi_{B2}; \end{gathered}$$

furthermore, there is the following relation between the numbers k_r and κ_r:

$$k_r = \left\lceil \frac{r\varphi_{C2}}{\psi} \right\rceil = \left\lceil \frac{r(\varphi_{B2} + \varphi_{BC}\psi)}{\psi} \right\rceil = \varphi_{BC} r + \left\lceil \frac{r\varphi_{B2}}{\psi} \right\rceil = \varphi_{BC} r + \kappa_r,$$

for all $r \in \{0, \ldots, \psi\}$.

Next, we use these data in rewriting both sums appearing in (94). If we adopt the convention that empty sums equal zero, then the first sum is given by

$$\begin{aligned}
\sum_{k=0}^{\varphi_{C2}-1} q^{k-\varphi_{BC}\left\lfloor \frac{k\psi}{\varphi_{C2}} \right\rfloor} \beta^{\left\lfloor \frac{k\psi}{\varphi_{C2}} \right\rfloor} &= \sum_{r=0}^{\psi-1} \sum_{k=k_r}^{k_{r+1}-1} q^{k-\varphi_{BC} r} \beta^r \\
&= \sum_r \beta^r \sum_{k=\varphi_{BC} r + \kappa_r}^{\varphi_{BC}(r+1)+\kappa_{r+1}-1} q^{k-\varphi_{BC} r} \\
&= \sum_r \beta^r \sum_{\kappa=\kappa_r}^{\varphi_{BC}+\kappa_{r+1}-1} q^{\kappa} \\
&\overset{(\star)}{=} \sum_r \beta^r \left(\sum_{\kappa=\kappa_r}^{\kappa_{r+1}-1} q^{\kappa} + q^{\kappa_{r+1}} \sum_{l=0}^{\varphi_{BC}-1} q^l \right) \\
&= \sum_r \beta^r \sum_\kappa q^\kappa + \frac{q^{\varphi_{BC}} - 1}{q-1} \sum_r \beta^r q^{\kappa_{r+1}}. \qquad (95)
\end{aligned}$$

Note that Equality $(\star)$ holds even if $\kappa_r = \kappa_{r+1}$ for some r. With Notation 4.2, the second sum can be written as

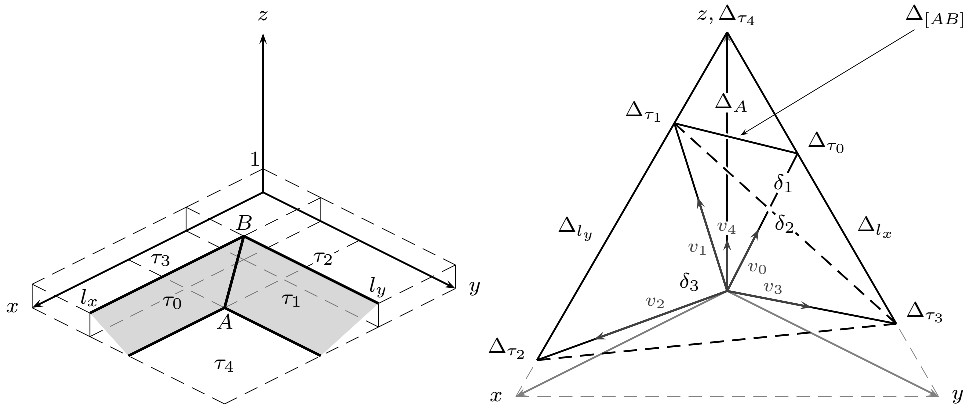
$$\sum_{\kappa=0}^{\varphi_{B2}-1} q^\kappa \beta^{\left\lfloor \frac{\kappa\psi}{\varphi_{B2}} \right\rfloor} = \sum_{r=0}^{\psi-1} \beta^r \sum_\kappa \left[\left\lfloor \frac{\kappa\psi}{\varphi_{B2}} \right\rfloor = r \right] q^\kappa = \sum_r \beta^r \sum_{\kappa=\kappa_r}^{\kappa_{r+1}-1} q^\kappa. \qquad (96)$$

Indeed, if there is no $\kappa \in \{0, \ldots, \varphi_{B2} - 1\}$ such that $\lfloor \kappa\psi/\varphi_{B2} \rfloor = r$, then $\kappa_r = \kappa_{r+1}$ and $\sum_{\kappa=\kappa_r}^{\kappa_{r+1}-1} q^\kappa = 0$, otherwise $\kappa_r < \kappa_{r+1}$ and $\kappa_r, \ldots, \kappa_{r+1} - 1$ are precisely the indices κ satisfying $\lfloor \kappa\psi/\varphi_{B2} \rfloor = r$.

From (95–96), it now follows that

$$\begin{aligned}
&(q-1)(\beta-1) \sum_{k=0}^{\varphi_{C2}-1} q^{k-\varphi_{BC}\left\lfloor \frac{k\psi}{\varphi_{C2}} \right\rfloor} \beta^{\left\lfloor \frac{k\psi}{\varphi_{C2}} \right\rfloor} + (q-1)(q^{\varphi_{BC}} - \beta) \sum_{\kappa=0}^{\varphi_{B2}-1} q^\kappa \beta^{\left\lfloor \frac{\kappa\psi}{\varphi_{B2}} \right\rfloor} \\
&= (q^{\varphi_{BC}} - 1) \left[(q-1) \sum_{r=0}^{\psi-1} \beta^r \sum_{\kappa=\kappa_r}^{\kappa_{r+1}-1} q^\kappa + (\beta - 1) \sum_{r=0}^{\psi-1} \beta^r q^{\kappa_{r+1}} \right] \\
&= (q^{\varphi_{BC}} - 1) \left[\sum_r \beta^r q^{\kappa_{r+1}} - \sum_r \beta^r q^{\kappa_r} + \sum_r \beta^{r+1} q^{\kappa_{r+1}} - \sum_r \beta^r q^{\kappa_{r+1}} \right] \\
&= (q^{\varphi_{BC}} - 1)(q^{\varphi_{B2}} \beta^\psi - 1).
\end{aligned}$$

Having verified that the calculations above make sense even if $\kappa_r = \kappa_{r+1}$ for some r, we achieve (94) and therefore conclude Case $n^* \nmid n$. This ends the proof of the main theorem in Case III.



(a) Non-compact B_1-facets τ_0 and τ_1, their subfaces and neighbor facets τ_2, τ_3, and τ_4

(b) Relevant cones associated to relevant faces of Γ_f

FIGURE 8. Case IV: the only facets contributing to s_0 are the non-compact B_1-facets τ_0 and τ_1

5. CASE IV: EXACTLY TWO FACETS OF Γ_f CONTRIBUTE TO s_0, AND THESE TWO FACETS ARE BOTH NON-COMPACT B_1-FACETS WITH RESPECT TO A SAME VARIABLE AND HAVE AN EDGE IN COMMON

5.1. **Figure and notations.** Let us assume that the two facets τ_0 and τ_1 contributing to s_0 are both B_1-facets with respect to the variable z. Note that τ_0 and τ_1 cannot be non-compact for the same variable unless they coincide. Therefore, we may assume that τ_0 is non-compact for x, while τ_1 is non-compact for y, and that τ_0 and τ_1 share their unique compact edge $[AB]$. Here $A(x_A, y_A, 0)$ and $B(x_B, y_B, 1) \in \mathbf{Z}^3_{\geqslant 0}$ denote the common vertices of τ_0 and τ_1 in the xy-plane and at 'height' one, respectively. The situation is shown in Figure 8.

If we put $\overrightarrow{AB}(x_B - x_A, y_B - y_A, 1) = (\alpha, \beta, 1)$ as usual, then $v_0(0, 1, -\beta)$ and $v_1(1, 0, -\alpha)$ are the unique primitive vectors in $\mathbf{Z}^3_{\geqslant 0}$ perpendicular to τ_0 and τ_1, respectively, while equations for the affine hulls of τ_0 and τ_1 are provided by

$$\operatorname{aff}(\tau_0) \leftrightarrow y - \beta z = y_A \qquad \text{and} \qquad \operatorname{aff}(\tau_1) \leftrightarrow x - \alpha z = x_A.$$

Necessarily, we have that $\alpha, \beta < 0$; i.e., $x_B < x_A$ and $y_B < y_A$. Given that the numerical data associated to τ_0 and τ_1 are $(m(v_0), \sigma(v_0)) = (y_A, 1 - \beta)$ and $(m(v_1), \sigma(v_1)) = (x_A, 1 - \alpha)$, respectively, we assume

$$\Re(s_0) = \frac{\beta - 1}{y_A} = \frac{\alpha - 1}{x_A} \qquad \text{and} \qquad \Im(s_0) = \frac{2n\pi}{\gcd(x_A, y_A) \log p} \qquad \text{for some } n \in \mathbf{Z}.$$

As indicated in Figure 8, we denote by τ_2 and τ_3 the non-compact facets of Γ_f sharing with τ_1 and τ_0, respectively, a half-line with endpoint B, and by τ_4 the facet lying in the xy-plane. Primitive vectors in $\mathbf{Z}^3_{\geqslant 0}$ perpendicular to τ_2, τ_3, τ_4 will be denoted

$$v_2(a_2, 0, c_2), \quad v_3(0, b_3, c_3), \quad v_4(0, 0, 1),$$

respectively, and equations for the affine supports of these facets are given by

$$\begin{aligned}
\operatorname{aff}(\tau_2) &\leftrightarrow a_2x + c_2z = m_2,\\
\operatorname{aff}(\tau_3) &\leftrightarrow b_3y + c_3z = m_3,\\
\operatorname{aff}(\tau_4) &\leftrightarrow \qquad\quad z = 0
\end{aligned}$$

for certain $m_2, m_3 \in \mathbf{Z}_{\geqslant 0}$. Finally, the numerical data for τ_2, τ_3, and τ_4 are (m_2, σ_2), (m_3, σ_3), and $(0,1)$, respectively, with $\sigma_2 = a_2 + c_2$ and $\sigma_3 = b_3 + c_3$.

5.2. **The candidate pole s_0 and the contributions to its residues.** Again we want to prove that s_0 is not a pole of Z_f^0. Since s_0 has expected order two as a candidate pole of Z_f^0, in order to do this, we will, as in Case III, show that

$$\begin{aligned}
R_2 &= \lim_{s\to s_0}\left(p^{1-\beta+y_As}-1\right)\left(p^{1-\alpha+x_As}-1\right)Z_f^0(s) \qquad \text{and}\\
R_1 &= \lim_{s\to s_0}\frac{d}{ds}\left[\left(p^{1-\beta+y_As}-1\right)\left(p^{1-\alpha+x_As}-1\right)Z_f^0(s)\right]
\end{aligned}$$

both equal zero.

In this case the (compact) faces contributing to s_0 are A, B, and $[AB]$; i.e., we may in the above expressions for R_2 and R_1 replace $Z_f^0(s)$ by

$$\sum_{\tau=A,B,[AB]} L_\tau(s)S(\Delta_\tau)(s).$$

Vertex A is exclusively contained in the facets τ_0, τ_1, and τ_4; its associated cone Δ_A is therefore simplicial. Vertex B, on the other hand, is contained in at least the facets τ_0, τ_1, τ_2, and τ_3; hence Δ_B is certainly not simplicial. However, if we consider the cones $\delta_1, \delta_2, \delta_3$ defined below as members of a simplicial subdivision of Δ_B, the relevant contributions to s_0 come from the simplicial cones

$$\begin{aligned}
\Delta_A = \operatorname{cone}(v_0, v_1, v_4), \qquad \delta_1 &= \operatorname{cone}(v_0, v_1, v_3), \qquad \Delta_{[AB]} = \operatorname{cone}(v_0, v_1).\\
\delta_2 &= \operatorname{cone}(v_1, v_3),\\
\delta_3 &= \operatorname{cone}(v_1, v_2, v_3),
\end{aligned}$$

This way we find, similarly to Case III, that R_2 and R_1 are explicitly given by

$$\begin{aligned}
R_2 &= L_A(s_0)\frac{\Sigma(\Delta_A)(s_0)}{p-1} + L_B(s_0)\frac{\Sigma(\delta_1)(s_0)}{p^{\sigma_3+m_3s_0}-1} + L_{[AB]}(s_0)\Sigma(\Delta_{[AB]})(s_0),\\
R_1 &= L_A'(s_0)\frac{\Sigma(\Delta_A)(s_0)}{p-1} + L_A(s_0)\frac{\Sigma(\Delta_A)'(s_0)}{p-1} + L_B'(s_0)\frac{\Sigma(\delta_1)(s_0)}{p^{\sigma_3+m_3s_0}-1}\\
&\quad + L_B(s_0)\frac{\Sigma(\delta_1)'(s_0)}{p^{\sigma_3+m_3s_0}-1} - L_B(s_0)\frac{m_3(\log p)p^{\sigma_3+m_3s_0}\Sigma(\delta_1)(s_0)}{\left(p^{\sigma_3+m_3s_0}-1\right)^2}\\
&\quad + L_B(s_0)\frac{y_A(\log p)\Sigma(\delta_2)(s_0)}{p^{\sigma_3+m_3s_0}-1} + L_B(s_0)\frac{y_A(\log p)\Sigma(\delta_3)(s_0)}{\left(p^{\sigma_2+m_2s_0}-1\right)\left(p^{\sigma_3+m_3s_0}-1\right)}\\
&\quad + L_{[AB]}'(s_0)\Sigma(\Delta_{[AB]})(s_0) + L_{[AB]}(s_0)\Sigma(\Delta_{[AB]})'(s_0).
\end{aligned}$$

5.3. **Towards simplified formulas for R_2 and R_1.**

5.3.1. *The factors $L_\tau(s_0)$ and $L'_\tau(s_0)$.* Since $N_A = N_B = 0$ and $N_{[AB]} = (p-1)^2$, we obtain

$$L_A(s_0) = L_B(s_0) = \left(\frac{p-1}{p}\right)^3, \qquad L'_A(s_0) = L'_B(s_0) = 0,$$
$$L_{[AB]}(s_0) = \left(\frac{p-1}{p}\right)^3 - \left(\frac{p-1}{p}\right)^2 \frac{p^{s_0}-1}{p^{s_0+1}-1}, \qquad \text{and}$$
$$L'_{[AB]}(s_0) = -(\log p)\left(\frac{p-1}{p}\right)^3 \frac{p^{s_0+1}}{\left(p^{s_0+1}-1\right)^2}.$$

5.3.2. *Cone multiplicities.* We calculate the multiplicities of the five contributing simplicial cones, as well as the multiplicities μ_x and μ_y of the cones associated to the non-compact edges $l_x = \tau_0 \cap \tau_3$ and $l_y = \tau_1 \cap \tau_2$ (see Figure 8):

$$\operatorname{mult}\Delta_A = \#H(v_0,v_1,v_4) = \left\|\begin{matrix} 0 & 1 & -\beta \\ 1 & 0 & -\alpha \\ 0 & 0 & 1 \end{matrix}\right\| = 1,$$
$$\mu_x = \operatorname{mult}\Delta_{l_x} = \#H(v_0,v_3) = \left\|\begin{matrix} 1 & -\beta \\ b_3 & c_3 \end{matrix}\right\| = -\left|\begin{matrix} 1 & -\beta \\ b_3 & c_3 \end{matrix}\right| = -\beta b_3 - c_3 > 0,$$
$$\mu_y = \operatorname{mult}\Delta_{l_y} = \#H(v_1,v_2) = \left\|\begin{matrix} 1 & -\alpha \\ a_2 & c_2 \end{matrix}\right\| = -\left|\begin{matrix} 1 & -\alpha \\ a_2 & c_2 \end{matrix}\right| = -\alpha a_2 - c_2 > 0,$$
$$\operatorname{mult}\delta_1 = \#H(v_0,v_1,v_3) = \left\|\begin{matrix} 0 & 1 & -\beta \\ 1 & 0 & -\alpha \\ 0 & b_3 & c_3 \end{matrix}\right\| = \left\|\begin{matrix} 1 & -\beta \\ b_3 & c_3 \end{matrix}\right\| = \mu_x,$$
$$\mu_3 = \operatorname{mult}\delta_3 = \#H(v_1,v_2,v_3) = \left\|\begin{matrix} 1 & 0 & -\alpha \\ a_2 & 0 & c_2 \\ 0 & b_3 & c_3 \end{matrix}\right\| = b_3\left\|\begin{matrix} 1 & -\alpha \\ a_2 & c_2 \end{matrix}\right\| = b_3\mu_y,$$
$$\operatorname{mult}\delta_2 = \#H(v_1,v_3) = \gcd(b_3, c_3, -\alpha b_3) = 1,$$
$$\operatorname{mult}\Delta_{[AB]} = \#H(v_0,v_1) = \gcd(1,-\beta,-\alpha) = 1.$$

5.3.3. *The sums $\Sigma(\cdot)(s_0)$ and $\Sigma(\cdot)'(s_0)$.* Because the corresponding multiplicities are one, we have that

$$\Sigma(\Delta_A)(s_0) = \Sigma(\delta_2)(s_0) = \Sigma(\Delta_{[AB]})(s_0) = 1 \qquad \text{and}$$
$$\Sigma(\Delta_A)'(s_0) = \Sigma(\Delta_{[AB]})'(s_0) = 0.$$

Furthermore, since $H(v_0,v_3) \subseteq H(v_0,v_1,v_3)$ and

$$\mu_x = \#H(v_0,v_3) = \#H(v_0,v_1,v_3),$$

we may put

$$H_x = H(v_0,v_3) = H(v_0,v_1,v_3),$$
$$\Sigma_x = \Sigma(\delta_1)(s_0) = \sum\nolimits_{h\in H_x} p^{\sigma(h)+m(h)s_0} = \sum\nolimits_{h\in H_x} p^{w\cdot h}, \qquad \text{and}$$
$$\Sigma'_x = \Sigma(\delta_1)'(s_0) = \frac{d}{ds}\left[\sum\nolimits_{h\in H_x} p^{\sigma(h)+m(h)s}\right]\Big|_{s=s_0} = (\log p)\sum\nolimits_{h\in H_x} m(h)p^{w\cdot h},$$

with $w = (1,1,1) + s_0(x_B, y_B, 1) \in \mathbf{C}^3$.

Finally we denote

$$H_y = H(v_1, v_2), \qquad H_3 = H(v_1, v_2, v_3), \qquad \text{and}$$
$$\Sigma_3 = \Sigma(\delta_3)(s_0) = \sum\nolimits_{h\in H_3} p^{\sigma(h)+m(h)s_0} = \sum\nolimits_{h\in H_3} p^{w\cdot h}.$$

5.3.4. *New formulas for the residues.* If we put

$$R_2 = \left(\frac{p-1}{p}\right)^3 R_2', \qquad R_1 = (\log p)\left(\frac{p-1}{p}\right)^3 R_1',$$
$$F_2 = p^{\sigma_2+m_2s_0} - 1, \qquad F_3 = p^{\sigma_3+m_3s_0} - 1, \qquad \text{and} \qquad q = p^{-s_0-1},$$

the observations above yield

$$R_2' = \frac{1}{1-q} + \frac{\Sigma_x}{F_3} \qquad \text{and} \tag{97}$$

$$R_1' = -\frac{q}{(1-q)^2} + \frac{\Sigma_x'}{(\log p)F_3} - \frac{m_3(F_3+1)\Sigma_x}{F_3^2} + \frac{y_A}{F_3} + \frac{y_A\Sigma_3}{F_2F_3}. \tag{98}$$

We shall prove that $R_2' = R_1' = 0$.

5.4. **Some vector identities and their consequences.** Given the coordinates of v_i; $i = 0, \dots, 3$; one easily checks that[30]

$$b_3v_0 - v_3 = (0, 0, \mu_x), \tag{99}$$
$$a_2v_1 - v_2 = (0, 0, \mu_y), \qquad \text{and} \tag{100}$$
$$-\mu_3v_0 + a_2\mu_xv_1 - \mu_xv_2 + \mu_yv_3 = (0, 0, 0). \tag{101}$$

Considering dot products with $w = (1, 1, 1) + s_0(x_B, y_B, 1)$, it follows from (99) and (100) that

$$\frac{-b_3(w\cdot v_0) + w\cdot v_3}{\mu_x} = \frac{-a_2(w\cdot v_1) + w\cdot v_2}{\mu_y} = -s_0 - 1, \tag{102}$$

whereas making the dot product with $B(x_B, y_B, 1)$ on both sides of (99) yields

$$y_Ab_3 - m_3 = \mu_x. \tag{103}$$

Other consequences of (99–101) include

$$\frac{-b_3}{\mu_x}v_0 + \frac{1}{\mu_x}v_3 = (0, 0, -1) \in \mathbf{Z}^3, \tag{104}$$
$$\frac{-a_2}{\mu_y}v_1 + \frac{1}{\mu_y}v_2 = (0, 0, -1) \in \mathbf{Z}^3, \qquad \text{and} \tag{105}$$
$$\frac{a_2\mu_x}{\mu_3}v_1 + \frac{-\mu_x}{\mu_3}v_2 + \frac{1}{b_3}v_3 = v_0 \in \mathbf{Z}^3. \tag{106}$$

[30] As in the previous cases, the first two identities arise from $(\operatorname{adj} M)M = (\det M)I$ for $M = \left(\begin{smallmatrix} 1 & -\beta \\ b_3 & c_3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & -\alpha \\ a_2 & c_2 \end{smallmatrix}\right)$ with $\det M = \mu_x, \mu_y$, respectively, while the third one follows immediately from the other two.

5.5. **Points of H_x, H_y, and H_3.** It follows from (104) that the μ_x points of H_x are given by

$$\left\{\frac{-jb_3}{\mu_x}\right\} v_0 + \frac{j}{\mu_x} v_3; \qquad j = 0, \ldots, \mu_x - 1; \tag{107}$$

while it follows from (105) that the μ_y points of $H_y = H(v_1, v_2)$ are

$$\left\{\frac{-ia_2}{\mu_y}\right\} v_1 + \frac{i}{\mu_y} v_2; \qquad i = 0, \ldots, \mu_y - 1.$$

Note that b_3 and a_2 are, as expected, coprime to μ_x and μ_y, respectively.[31]

If we consider $H_3 = H(v_1, v_2, v_3)$ in the usual way as an additive group with subgroup[32] $H_y = H_y + H(v_1, v_3)$ of index b_3, then we see from (105) and (106) that the points

$$\left\{\frac{-a_2\{-k\mu_x\}_{b_3}}{\mu_3}\right\} v_1 + \frac{\{-k\mu_x\}_{b_3}}{\mu_3} v_2 + \frac{k}{b_3} v_3 \in H_3; \qquad k = 0, \ldots, b_3 - 1;$$

can serve as representatives for the b_3 cosets of H_y in H_3. Hence a complete list of the $\mu_3 = b_3\mu_y$ points of H_3 is provided by

$$\begin{aligned} &\left\{\frac{-a_2(ib_3 + \{-k\mu_x\}_{b_3})}{\mu_3}\right\} v_1 + \frac{ib_3 + \{-k\mu_x\}_{b_3}}{\mu_3} v_2 + \frac{k}{b_3} v_3; \\ &\qquad\qquad i = 0, \ldots, \mu_y - 1; \quad k = 0, \ldots, b_3 - 1. \end{aligned} \tag{108}$$

These descriptions should allow us to find expressions for Σ_x, Σ'_x, and Σ_3 in the next subsection.

5.6. **Formulas for Σ_x, Σ'_x, and Σ_3.** If for $h \in H_x = H(v_0, v_3)$, we denote by (h_0, h_3) the coordinates of h with respect to the basis (v_0, v_3), then by (102), (107), and $p^{w\cdot v_0} = 1$, we have

$$\Sigma_x = \sum_{h\in H_x} p^{w\cdot h} = \sum_{j=0}^{\mu_x - 1} \left(p^{\frac{-b_3(w\cdot v_0) + w\cdot v_3}{\mu_x}}\right)^j = \sum_j q^j = \frac{F_3}{q-1}, \tag{109}$$

whereas

$$\begin{aligned} \frac{\Sigma'_x}{\log p} &= \sum_{h\in H_x} m(h) p^{w\cdot h} \\ &= \sum_h (h_0 y_A + h_3 m_3) p^{w\cdot h} \\ &= \sum_{j=0}^{\mu_x - 1} \left(y_A \left\{\frac{-jb_3}{\mu_x}\right\} + m_3 \frac{j}{\mu_x}\right) \left(p^{\frac{-b_3(w\cdot v_0) + w\cdot v_3}{\mu_x}}\right)^j \\ &= y_A \sum_j \left\{\frac{-jb_3}{\mu_x}\right\} q^j + \frac{m_3}{\mu_x} \sum_j j q^j. \end{aligned} \tag{110}$$

If $\mu_x = 1$, then clearly $\Sigma'_x = 0$. Let us find an expression for Σ'_x in the complementary case. So from now on assume that $\mu_x > 1$. Write b_3 as $b_3 = t\mu_x + \overline{b_3}$ with

[31]This follows from $\mu_x = -\beta b_3 - c_3$, $\mu_y = -\alpha a_2 - c_2$, and the primitivity of v_2 and v_3.

[32]Recall that $H(v_1, v_3)$ is the trivial subgroup of H_3.

$t \in \mathbf{Z}_{\geqslant 0}$ and $\overline{b_3} = \{b_3\}_{\mu_x} \in \{1, \ldots, \mu_x - 1\}$; note that by the coprimality of b_3 and μ_x, we have $\gcd(\overline{b_3}, \mu_x) = 1$ and hence $\overline{b_3} \neq 0$. Furthermore, put

$$j_{\overline{k}} = \min\left\{ j \in \mathbf{Z}_{\geqslant 0} \;\middle|\; \left\lfloor \frac{j\overline{b_3}}{\mu_x} \right\rfloor = \overline{k} \right\} = \left\lceil \frac{\overline{k}\mu_x}{\overline{b_3}} \right\rceil; \qquad \overline{k} = 0, \ldots, \overline{b_3}; \tag{111}$$

yielding

$$0 = j_0 < j_1 < \cdots < j_{\overline{b_3}-1} < j_{\overline{b_3}} = \mu_x.$$

Then, proceeding as in Case III (Subsection 4.15), we write (110) as

$$\begin{aligned}
&\frac{\Sigma_x'}{\log p}\\
&= y_A \sum_{j=0}^{\mu_x - 1} \left\{ \frac{-j\overline{b_3}}{\mu_x} \right\} q^j + \frac{m_3[q^{\mu_x}(\mu_x q - \mu_x - q) + q]}{\mu_x(q-1)^2}\\
&= \frac{y_A}{1-q}\left(\sum_{\overline{k}=0}^{\overline{b_3}-1} q^{j_{\overline{k}}} - \frac{\overline{b_3}F_3}{\mu_x(1-q^{-1})} \right) - y_A + \frac{m_3}{1-q}\left(-(F_3+1) + \frac{F_3}{\mu_x(1-q^{-1})} \right)\\
&= \frac{F_3}{1-q}\left(\frac{y_A}{F_3} \sum_{\overline{k}} q^{j_{\overline{k}}} - \frac{y_A\overline{b_3} - m_3}{\mu_x(1-q^{-1})} - \frac{m_3(F_3+1)}{F_3} \right) - y_A.
\end{aligned}$$

Finally, if we use that $y_A b_3 - m_3 = \mu_x$ (cfr. (103)) and $b_3 = t\mu_x + \overline{b_3}$, we obtain (112)

$$\frac{\Sigma_x'}{(\log p)F_3} = \frac{1}{1-q}\left(\frac{y_A}{F_3} \sum_{\overline{k}=0}^{\overline{b_3}-1} q^{j_{\overline{k}}} + \frac{y_A t - 1}{1-q^{-1}} - \frac{m_3(F_3+1)}{F_3} \right) - \frac{y_A}{F_3} \qquad (\text{if } \mu_x > 1).$$

Let us now calculate Σ_3. Using (102, 108) and $p^{w\cdot v_0} = p^{w\cdot v_1} = 1$, we find

$$\begin{aligned}
\Sigma_3 &= \sum_{h\in H_3} p^{w\cdot h}\\
&= \sum_{h} p^{h_1(w\cdot v_1)+h_2(w\cdot v_2)+h_3(w\cdot v_3)}\\
&= \sum_{i=0}^{\mu_y-1}\sum_{k=0}^{b_3-1} p^{\frac{-a_2\left(ib_3+\{-k\mu_x\}_{b_3}\right)}{\mu_3}(w\cdot v_1)+\frac{ib_3+\{-k\mu_x\}_{b_3}}{\mu_3}(w\cdot v_2)+\frac{k}{b_3}(w\cdot v_3)}\\
&= \sum_{i}\left(p^{\frac{-a_2(w\cdot v_1)+w\cdot v_2}{\mu_y}} \right)^i \sum_k p^{\frac{-a_2(w\cdot v_1)+w\cdot v_2}{\mu_y}\left\{\frac{-k\mu_x}{b_3}\right\}+\frac{-b_3(w\cdot v_0)+w\cdot v_3}{\mu_x}\frac{k\mu_x}{b_3}}\\
&= \sum_i q^i \sum_k p^{(-s_0-1)\left(\left\{\frac{-k\mu_x}{b_3}\right\}+\frac{k\mu_x}{b_3}\right)}.
\end{aligned}$$

Since μ_x and b_3 are coprime, one has $k\mu_x/b_3 \notin \mathbf{Z}$ and $\{-k\mu_x/b_3\} = 1 - \{k\mu_x/b_3\}$ for k not a multiple of b_3. Hence

$$\left\{ -\frac{k\mu_x}{b_3} \right\} + \frac{k\mu_x}{b_3} = \begin{cases} 0, & \text{if } k = 0;\\ 1 + \left\lfloor \frac{k\mu_x}{b_3} \right\rfloor \in \mathbf{Z}, & \text{if } k \in \{1, \ldots, b_3 - 1\}; \end{cases}$$

and we obtain

$$\Sigma_3 = \frac{F_2}{q-1}\left(1 + q\sum_{k=1}^{b_3-1} q^{\left\lfloor \frac{k\mu_x}{b_3} \right\rfloor} \right), \tag{113}$$

with the understanding that the empty sum equals zero in case $b_3 = 1$.

5.7. **Proof of $R_2' = R_1' = 0$.** As it follows immediately from (97) and (109) that $R_2' = 0$, we can further focus on R_1'. Let us first assume that $\mu_x = 1$. In this case, we found that $\Sigma_x' = 0$, while it follows from (113) that

$$\Sigma_3 = \frac{F_2}{q-1}(1 + (b_3 - 1)q);$$

furthermore, note that $F_3 = q - 1$ and hence $\Sigma_x = 1$, while $y_A b_3 - m_3 = 1$ by (103). With these observations, (98) easily yields $R_1' = 0$.

From now on, suppose that $\mu_x > 1$ and thus that $\overline{b_3} > 0$. If we then fill in (109, 112, 113) in (98), one sees that proving $R_1' = 0$ eventually boils down to proving that

$$\sum_{k=1}^{b_3-1} q^{\left\lfloor \frac{k\mu_x}{b_3} \right\rfloor} = \sum_{\overline{k}=1}^{\overline{b_3}-1} q^{j_{\overline{k}}-1} + t\frac{F_3}{q-1}, \tag{114}$$

whereby the sum over $\overline{k}$ is again understood to be zero if $\overline{b_3} = 1$. Let us do this now.

Recall that $b_3 = t\mu_x + \overline{b_3}$ with $t \in \mathbf{Z}_{\geqslant 0}$ and $\overline{b_3} \in \{1, \ldots, \mu_x - 1\}$. So if $t = 0$, we have $b_3 = \overline{b_3}$, and by (111) and the coprimality of $\overline{b_3}$ and μ_x, we then find

$$\sum_{k=1}^{b_3-1} q^{\left\lfloor \frac{k\mu_x}{b_3} \right\rfloor} = \sum_{\overline{k}=1}^{\overline{b_3}-1} q^{\left\lfloor \frac{\overline{k}\mu_x}{\overline{b_3}} \right\rfloor} = \sum_{\overline{k}} q^{\left\lceil \frac{\overline{k}\mu_x}{\overline{b_3}} \right\rceil - 1} = \sum_{\overline{k}} q^{j_{\overline{k}}-1},$$

which agrees with (114) for $t = 0$.

In what follows, we assume that $t > 0$ and hence that $b_3 > \mu_x$. Define the numbers

$$k_j = \min\left\{ k \in \mathbf{Z}_{\geqslant 0} \;\middle|\; \left\lfloor \frac{k\mu_x}{b_3} \right\rfloor = j \right\} = \left\lceil \frac{jb_3}{\mu_x} \right\rceil; \qquad j = 0, \ldots, \mu_x;$$

and note that

$$0 = k_0 < k_1 < \cdots < k_{\mu_x - 1} < k_{\mu_x} = b_3.$$

This gives rise to

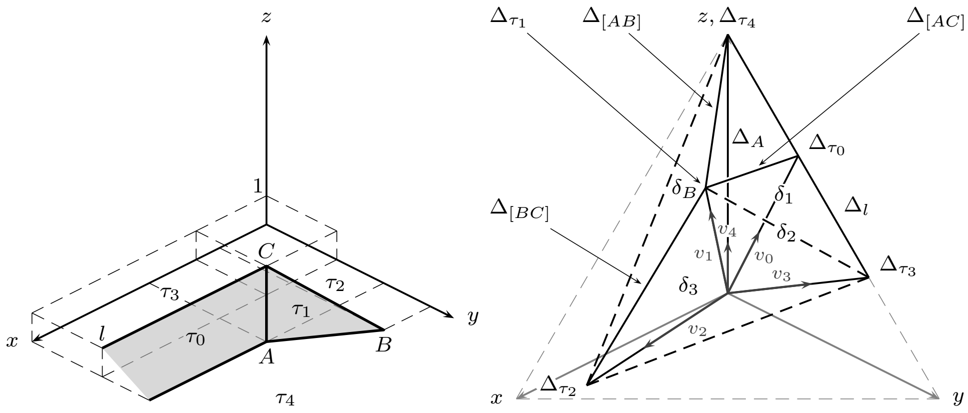
$$\sum_{k=1}^{b_3-1} q^{\left\lfloor \frac{k\mu_x}{b_3} \right\rfloor} = \sum_{j=0}^{\mu_x-1} \sum_{k=k_j}^{k_{j+1}-1} q^j - 1 = \sum_{\overline{k}=0}^{\overline{b_3}-1} \sum_{j=j_{\overline{k}}}^{j_{\overline{k}+1}-1} (k_{j+1} - k_j) q^j - 1.$$

Finally, observe that

$$k_j = \left\lceil \frac{jb_3}{\mu_x} \right\rceil = \left\lceil \frac{j(t\mu_x + \overline{b_3})}{\mu_x} \right\rceil = jt + \left\lceil \frac{j\overline{b_3}}{\mu_x} \right\rceil = jt + \left\lfloor \frac{j\overline{b_3}}{\mu_x} \right\rfloor + 1 - [j = 0] - [j = \mu_x]$$

for $j \in \{0, \ldots, \mu_x\}$; hence for $0 \leqslant \overline{k} \leqslant \overline{b_3} - 1$ and $j_{\overline{k}} \leqslant j \leqslant j_{\overline{k}+1} - 1$, we have

$$\begin{aligned} &k_{j+1} - k_j \\ &= \big((j+1)t + (\overline{k} + [j + 1 = j_{\overline{k}+1}]) + 1 - [j + 1 = \mu_x]\big) - \big(jt + \overline{k} + 1 - [j = 0]\big) \\ &= t + [j = j_{\overline{k}+1} - 1] + [j = 0] - [j = \mu_x - 1]. \end{aligned}$$



(a) Non-compact B_1-facet τ_0, B_1-simplex τ_1, their subfaces and neighbor facets τ_2, τ_3, τ_4

(b) Relevant cones associated to relevant faces of Γ_f

FIGURE 9. Case V: the only facets contributing to s_0 are the non-compact B_1-facet τ_0 and the B_1-simplex τ_1

Therefore,

$$\begin{aligned}
\sum_{k=1}^{b_3-1} q^{\left\lfloor \frac{k\mu_x}{b_3} \right\rfloor} &= \sum_{\overline{k}=0}^{\overline{b_3}-1} \sum_{j=j_{\overline{k}}}^{j_{\overline{k}+1}-1} (k_{j+1}-k_j) q^j - 1 \\
&= \sum_{\overline{k}} \sum_{j} \big(t + [j = j_{\overline{k}+1}-1] + [j=0] - [j = \mu_x - 1]\big) q^j - 1 \\
&= t \sum_{j=0}^{\mu_x-1} q^j + \sum_{\overline{k}=0}^{\overline{b_3}-1} q^{j_{\overline{k}+1}-1} + q^0 - q^{\mu_x-1} - 1 \\
&= \sum_{\overline{k}=1}^{\overline{b_3}-1} q^{j_{\overline{k}}-1} + t\frac{F_3}{q-1},
\end{aligned}$$

which agrees with (114). This concludes Case IV.

6. Case V: exactly two facets of Γ_f contribute to s_0; one of them is a non-compact B_1-facet, the other one a B_1-simplex; these facets are B_1 with respect to a same variable and have an edge in common

6.1. **Figure and notations.** We assume that the two facets τ_0 and τ_1 contributing to s_0 are both B_1-facets with respect to the variable z. Let τ_0 be non-compact, say for the variable x, and let τ_1 be a B_1-simplex. Facet τ_0 shares its unique compact edge $[AC]$ with τ_1. We denote the vertices of τ_0 and τ_1 and their coordinates by

$$A(x_A, y_A, 0), \quad B(x_B, y_B, 0), \quad C(x_C, y_C, 1)$$

and the neighbor facets of τ_0 and τ_1 by τ_2, τ_3, τ_4 as indicated in Figure 9.

Let us put

$$\begin{aligned}
&\overrightarrow{AC}(x_C - x_A, y_C - y_A, 1) = (\alpha_A, \beta_A, 1),\\
&\overrightarrow{BC}(x_C - x_B, y_C - y_B, 1) = (\alpha_B, \beta_B, 1),\\
&\text{and} \qquad \varphi_{AB} = \gcd(x_B - x_A, y_B - y_A)
\end{aligned}$$

as before. The unique primitive vector $v_0 \in \mathbf{Z}^3_{\geqslant 0}$ perpendicular to τ_0 is given by $v_0(0, 1, -\beta_A)$; such vectors for the other relevant facets $\tau_1, \tau_2, \tau_3, \tau_4$ will be denoted

$$v_1(a_1, b_1, c_1), \quad v_2(a_2, b_2, c_2), \quad v_3(0, b_3, c_3), \quad v_4(0, 0, 1),$$

respectively. Equations for the affine supports of τ_i; $i = 0, \dots, 4$; are given by

$$\begin{aligned}
\operatorname{aff}(\tau_0) &\leftrightarrow & y - \beta_A z &= y_A,\\
\operatorname{aff}(\tau_1) &\leftrightarrow a_1 x + & b_1 y + \ c_1 z &= m_1,\\
\operatorname{aff}(\tau_2) &\leftrightarrow a_2 x + & b_2 y + \ c_2 z &= m_2,\\
\operatorname{aff}(\tau_3) &\leftrightarrow & b_3 y + \ c_3 z &= m_3,\\
\operatorname{aff}(\tau_4) &\leftrightarrow & z &= 0
\end{aligned}$$

for certain $m_1, m_2, m_3 \in \mathbf{Z}_{\geqslant 0}$, and to these facets we associate the respective numerical data

$$(y_A, 1 - \beta_A), \quad (m_1, \sigma_1), \quad (m_2, \sigma_2), \quad (m_3, \sigma_3), \quad (0, 1),$$

with $\sigma_i = a_i + b_i + c_i$; $i = 1, 2$; and $\sigma_3 = b_3 + c_3$.

Since we assume that τ_0 and τ_1 both contribute to the candidate pole s_0, we have that $p^{1-\beta_A+y_A s_0} = p^{\sigma_1+m_1 s_0} = 1$; hence

$$\Re(s_0) = \frac{\beta_A - 1}{y_A} = -\frac{\sigma_1}{m_1} \quad \text{and} \quad \Im(s_0) = \frac{2n\pi}{\gcd(y_A, m_1)\log p} \quad \text{for some } n \in \mathbf{Z}.$$

6.2. **Contributions to the candidate pole s_0.** The goal of this section is again to show that both

$$\begin{aligned}
R_2 &= \lim_{s\to s_0} \left(p^{1-\beta_A+y_A s} - 1\right)\left(p^{\sigma_1+m_1 s} - 1\right) Z_f^0(s) \qquad \text{and}\\
R_1 &= \lim_{s\to s_0} \frac{d}{ds}\left[\left(p^{1-\beta_A+y_A s} - 1\right)\left(p^{\sigma_1+m_1 s} - 1\right) Z_f^0(s)\right]
\end{aligned}$$

equal zero. The compact faces of Γ_f contributing to s_0 are again the (seven) compact subfaces $A, B, C, [AB], [AC], [BC]$, and τ_1 of the two contributing facets τ_0 and τ_1. Only three of them also contribute to the 'residue' R_2: A, C, and $[AC]$.

If we consider the nine simplicial cones

$$\begin{aligned}
\Delta_{\tau_1} &= \operatorname{cone}(v_1), & \delta_1 &= \operatorname{cone}(v_0, v_1, v_3), & \Delta_{[AB]} &= \operatorname{cone}(v_1, v_4),\\
\Delta_A &= \operatorname{cone}(v_0, v_1, v_4), & \delta_2 &= \operatorname{cone}(v_1, v_3), & \Delta_{[AC]} &= \operatorname{cone}(v_0, v_1),\\
\delta_B &= \operatorname{cone}(v_1, v_2, v_4), & \delta_3 &= \operatorname{cone}(v_1, v_2, v_3), & \Delta_{[BC]} &= \operatorname{cone}(v_1, v_2),
\end{aligned}$$

the same approach as in Cases III and IV leads to the following expressions for R_2 and R_1:

$$
\begin{aligned}
R_2 &= L_A(s_0)\frac{\Sigma(\Delta_A)(s_0)}{p-1} + L_C(s_0)\frac{\Sigma(\delta_1)(s_0)}{p^{\sigma_3+m_3s_0}-1} + L_{[AC]}(s_0)\Sigma(\Delta_{[AC]})(s_0),\\
R_1 &= L'_A(s_0)\frac{\Sigma(\Delta_A)(s_0)}{p-1} + L_A(s_0)\frac{\Sigma(\Delta_A)'(s_0)}{p-1} + L_B(s_0)\frac{y_A(\log p)\Sigma(\delta_B)(s_0)}{\left(p^{\sigma_2+m_2s_0}-1\right)(p-1)}\\
&\quad +L'_C(s_0)\frac{\Sigma(\delta_1)(s_0)}{p^{\sigma_3+m_3s_0}-1} + L_C(s_0)\frac{\Sigma(\delta_1)'(s_0)}{p^{\sigma_3+m_3s_0}-1}\\
&\quad -L_C(s_0)\frac{m_3(\log p)p^{\sigma_3+m_3s_0}\Sigma(\delta_1)(s_0)}{\left(p^{\sigma_3+m_3s_0}-1\right)^2} + L_C(s_0)\frac{y_A(\log p)\Sigma(\delta_2)(s_0)}{p^{\sigma_3+m_3s_0}-1}\\
&\quad +L_C(s_0)\frac{y_A(\log p)\Sigma(\delta_3)(s_0)}{\left(p^{\sigma_2+m_2s_0}-1\right)\left(p^{\sigma_3+m_3s_0}-1\right)} + L_{[AB]}(s_0)\frac{y_A(\log p)\Sigma(\Delta_{[AB]})(s_0)}{p-1}\\
&\quad +L'_{[AC]}(s_0)\Sigma(\Delta_{[AC]})(s_0) + L_{[AC]}(s_0)\Sigma(\Delta_{[AC]})'(s_0)\\
&\quad +L_{[BC]}(s_0)\frac{y_A(\log p)\Sigma(\Delta_{[BC]})(s_0)}{p^{\sigma_2+m_2s_0}-1} + L_{\tau_1}(s_0)y_A(\log p)\Sigma(\Delta_{\tau_1})(s_0).
\end{aligned}
$$

6.3. Towards simplified formulas for R_2 and R_1.

6.3.1. *The factors $L_\tau(s_0)$ and $L'_\tau(s_0)$.* In the usual way we obtain

$$
\begin{aligned}
L_A(s_0) = L_B(s_0) = L_C(s_0) &= \left(\frac{p-1}{p}\right)^3, \qquad L'_A(s_0) = L'_C(s_0) = 0,\\
L_{[AB]}(s_0) &= \left(\frac{p-1}{p}\right)^3 - \frac{(p-1)N}{p^2}\frac{p^{s_0}-1}{p^{s_0+1}-1},\\
L_{[AC]}(s_0) = L_{[BC]}(s_0) &= \left(\frac{p-1}{p}\right)^3 - \left(\frac{p-1}{p}\right)^2\frac{p^{s_0}-1}{p^{s_0+1}-1},\\
L'_{[AC]}(s_0) &= -(\log p)\left(\frac{p-1}{p}\right)^3\frac{p^{s_0+1}}{\left(p^{s_0+1}-1\right)^2},\\
\text{and} \qquad L_{\tau_1}(s_0) &= \left(\frac{p-1}{p}\right)^3 - \frac{(p-1)^2-N}{p^2}\frac{p^{s_0}-1}{p^{s_0+1}-1},
\end{aligned}
$$

with

$$N = \#\left\{(x,y)\in(\mathbf{F}_p^\times)^2 \mid \overline{f_{[AB]}}(x,y)=0\right\}.$$

6.3.2. *Cone multiplicities.* Let us investigate the multiplicities of the nine contributing simplicial cones. As we did before, we shall also consider the multiplicities μ_l and μ'_1 of the respective simplicial cones Δ_l and $\delta'_1 = \operatorname{cone}(v_0, v_1, v_2)$; the first cone is the cone associated to the half-line $l = \tau_0 \cap \tau_3$ (see Figure 9), while the second one is a simplicial subcone of Δ_C that could have been chosen as a member of an alternative subdivision of Δ_C. Proceeding as in the previous cases, we find

$$\operatorname{mult}\Delta_{[AB]} = \operatorname{mult}\Delta_{\tau_1} = 1,$$

$$\mu_A = \operatorname{mult}\Delta_A = \#H(v_0,v_1,v_4) = \operatorname{mult}\Delta_{[AC]} = \#H(v_0,v_1) = a_1,$$

$$\mu_B = \operatorname{mult}\delta_B = \#H(v_1,v_2,v_4) = \operatorname{mult}\Delta_{[BC]} = \#H(v_1,v_2) = -\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix},$$

$$\mu_l = \operatorname{mult}\Delta_l = \#H(v_0,v_3) = -\begin{vmatrix} 1 & -\beta_A \\ b_3 & c_3 \end{vmatrix},$$

$$\mu_1 = \operatorname{mult}\delta_1 = \#H(v_0,v_1,v_3) = \begin{vmatrix} 0 & 1 & -\beta_A \\ a_1 & b_1 & c_1 \\ 0 & b_3 & c_3 \end{vmatrix} = -a_1\begin{vmatrix} 1 & -\beta_A \\ b_3 & c_3 \end{vmatrix} = \mu_A\mu_l,$$

$$\mu_1' = \operatorname{mult}\delta_1' = \#H(v_0,v_1,v_2) = \begin{vmatrix} 0 & 1 & -\beta_A \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \mu_A\mu_B\varphi_{AB}.$$

We observe that

$$\mu_1 = \#H(v_0,v_1,v_3) = \mu_A\mu_l = \#H(v_0,v_1)\#H(v_0,v_3);$$

i.e., the factor $\varphi_{Al} = \mu_1/\mu_A\mu_l$ equals one. Theorem 1.7(v) now asserts that

$$\mu_2 = \operatorname{mult}\delta_2 = \#H(v_1,v_3) = \gcd(\mu_A,\mu_l).$$

For

$$\mu_3 = \operatorname{mult}\delta_3 = \#H(v_1,v_2,v_3) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{vmatrix} > 0,$$

finally, we obtain in a similar way as in Case III that

$$\begin{aligned}
\mu_3 &= -b_3\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + c_3\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \\
&= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}(b_3\beta_B + c_3) \\
&= -\mu_B\left(v_3\cdot\overrightarrow{BC}\right) \\
&= \mu_B\left(v_3\cdot\overrightarrow{AB} - v_3\cdot\overrightarrow{AC}\right) \\
&= \mu_B\big(v_3\cdot\varphi_{AB}(-b_1,a_1,0) - v_3\cdot(\alpha_A,\beta_A,1)\big) \\
&= \mu_B\big(\varphi_{AB}\Psi + \mu_l\big) \\
&= \mu_B\mu_2\varphi_{B2},
\end{aligned}$$

whereby

$$\Psi = a_1b_3 = b_3\mu_A, \qquad \varphi_{B2} = \varphi_{AB}\psi + \mu_l',$$

$$\psi = \frac{\Psi}{\mu_2} = b_3\mu_A' \in \mathbf{Z}_{>0}, \qquad \mu_A' = \frac{\mu_A}{\mu_2} \in \mathbf{Z}_{>0}, \qquad \text{and} \qquad \mu_l' = \frac{\mu_l}{\mu_2} \in \mathbf{Z}_{>0}.$$

Note that the coprimality of b_3 and c_3 implies the coprimality of b_3 and μ_l. Hence

$$\mu_2 = \gcd(\mu_A,\mu_l) = \gcd(\Psi,\mu_l)$$

and $\gcd(\psi,\mu_l') = \gcd(\psi,\varphi_{B2}) = 1$.

6.3.3. *The sums* $\Sigma(\cdot)(s_0)$ *and* $\Sigma(\cdot)'(s_0)$. We have of course that $\Sigma(\Delta_{[AB]})(s_0) = \Sigma(\Delta_{\tau_1})(s_0) = 1$. As usual, we denote

$$\begin{aligned} H_A &= H(v_0,v_1,v_4) = H(v_0,v_1), & H_1 &= H(v_0,v_1,v_3),\\ H_B &= H(v_1,v_2,v_4) = H(v_1,v_2), & H_2 &= H(v_1,v_3),\\ H_l &= H(v_0,v_3), & H_3 &= H(v_1,v_2,v_3), \end{aligned}$$

and $w = (1,1,1) + s_0(x_C, y_C, 1) \in \mathbf{C}^3$, yielding

$$\begin{aligned} \Sigma_A = \Sigma(\Delta_A)(s_0) &= \Sigma(\Delta_{[AC]})(s_0) = \sum\nolimits_{h\in H_A} p^{w\cdot h};\\ \Sigma_B = \Sigma(\delta_B)(s_0) &= \Sigma(\Delta_{[BC]})(s_0) = \sum\nolimits_{h\in H_B} p^{w\cdot h};\\ \Sigma_A' = \Sigma(\Delta_A)'(s_0) &= \Sigma(\Delta_{[AC]})'(s_0) = (\log p)\sum\nolimits_{h\in H_A} m(h)p^{w\cdot h};\\ \Sigma_l &= \Sigma(\Delta_l)(s_0) = \sum\nolimits_{h\in H_l} p^{w\cdot h};\\ \Sigma_i &= \Sigma(\delta_i)(s_0) = \sum\nolimits_{h\in H_i} p^{w\cdot h}; \qquad i=1,2,3;\\ \Sigma_1' &= \Sigma(\delta_1)'(s_0) = (\log p)\sum\nolimits_{h\in H_1} m(h)p^{w\cdot h}. \end{aligned}$$

6.3.4. *New formulas for the residues.* Let us put

$$R_2 = \left(\frac{p-1}{p}\right)^3 R_2', \qquad R_1 = (\log p)\left(\frac{p-1}{p}\right)^3 R_1',$$
$$F_2 = p^{\sigma_2+m_2s_0} - 1, \qquad F_3 = p^{\sigma_3+m_3s_0} - 1, \qquad \text{and} \qquad q = p^{-s_0-1}.$$

Our findings so far lead to the following expressions for R_2' and R_1':

$$R_2' = \frac{\Sigma_A}{1-q} + \frac{\Sigma_1}{F_3}, \tag{115}$$

$$\begin{aligned} R_1' = \frac{1}{1-q}\left(\frac{\Sigma_A'}{\log p} - \frac{\Sigma_A}{q^{-1}-1} + \frac{y_A\Sigma_B}{F_2} + y_A\right)\\ + \frac{\Sigma_1'}{(\log p)F_3} - \frac{m_3(F_3+1)\Sigma_1}{F_3^2} + \frac{y_A\Sigma_2}{F_3} + \frac{y_A\Sigma_3}{F_2F_3}. \end{aligned} \tag{116}$$

We prove that $R_2' = R_1' = 0$.

6.4. Investigation of the sums $\Sigma_\bullet$ and $\Sigma_\bullet'$.

6.4.1. *Vector identities and consequences.* The identities that will be useful to us in this case are

$$b_3v_0 - v_3 = (0,0,\mu_l), \tag{117}$$

$$-\mu_Bv_0 + a_2v_1 - \mu_Av_2 = (0,0,\mu_1'), \qquad \text{and} \tag{118}$$
$$\Theta v_1 - \Psi v_2 - \mu_Bv_3 = (0,0,\mu_3),$$

whereby $\Theta = a_2b_3$ and $\Psi = a_1b_3$. These give rise to

$$y_Ab_3 - m_3 = \mu_l \tag{119}$$

and to

$$-b_3(w \cdot v_0) + w \cdot v_3 = \mu_l(-s_0 - 1), \tag{120}$$

$$\mu_B(w \cdot v_0) - a_2(w \cdot v_1) + \mu_A(w \cdot v_2) = \mu_1'(-s_0 - 1), \qquad \text{and} \tag{121}$$

$$-\Theta(w \cdot v_1) + \Psi(w \cdot v_2) + \mu_B(w \cdot v_3) = \mu_3(-s_0 - 1). \tag{122}$$

Moreover, they show that

$$\frac{-b_3}{\mu_l} v_0 + \frac{1}{\mu_l} v_3 \in \mathbf{Z}^3 \quad \text{and} \quad \frac{-\Theta}{\mu_3} v_1 + \frac{\psi}{\mu_B \varphi_{B2}} v_2 + \frac{1}{\mu_2 \varphi_{B2}} v_3 \in \mathbf{Z}^3. \tag{123}$$

(Recall that $\mu_3 = \mu_B \mu_2 \varphi_{B2}$ and $\Psi = \psi \mu_2$.)

6.4.2. *Points of* H_A, H_B, H_l, H_1, H_2, *and* H_3. The μ_A points of H_A are given by

$$\left\{ \frac{i\xi_A}{\mu_A} \right\} v_0 + \frac{i}{\mu_A} v_1; \qquad i = 0, \ldots, \mu_A - 1;$$

or, alternatively, by

$$\frac{i}{\mu_A} v_0 + \left\{ \frac{i\xi_A'}{\mu_A} \right\} v_1; \qquad i = 0, \ldots, \mu_A - 1; \tag{124}$$

for certain $\xi_A, \xi_A' \in \{0, \ldots, \mu_A - 1\}$ with $\xi_A \xi_A' \equiv 1 \bmod \mu_A$. By (123), the μ_l points of H_l are

$$\left\{ \frac{-jb_3}{\mu_l} \right\} v_0 + \frac{j}{\mu_l} v_3; \qquad j = 0, \ldots, \mu_l - 1; \tag{125}$$

while those of H_B and H_2 are given by

$$\left\{ \frac{i\xi_B}{\mu_B} \right\} v_1 + \frac{i}{\mu_B} v_2; \qquad i = 0, \ldots, \mu_B - 1; \tag{126}$$

and by

$$\left\{ \frac{j\xi_2}{\mu_2} \right\} v_1 + \frac{j}{\mu_2} v_3; \qquad j = 0, \ldots, \mu_2 - 1; \tag{127}$$

respectively, for unique $\xi_B \in \{0, \ldots, \mu_B - 1\}$ and $\xi_2 \in \{0, \ldots, \mu_2 - 1\}$, coprime to μ_B and μ_2, respectively.

Since $\mu_1 = \mu_A \mu_l$, the description of the μ_1 points of H_1 is rather easy:

$$\left\{ \frac{i\xi_A \mu_l - jb_3 \mu_A}{\mu_1} \right\} v_0 + \frac{i}{\mu_A} v_1 + \frac{j}{\mu_l} v_3; \qquad i = 0, \ldots, \mu_A - 1; \quad j = 0, \ldots, \mu_l - 1. \tag{128}$$

Based on (126–127) and (123), we find in exactly the same way as in Case III (Paragraph 4.17.2) a complete list of the $\mu_3 = \mu_B \mu_2 \varphi_{B2}$ points of H_3:

$$\begin{aligned} &\left\{ \frac{(i - \lfloor k\psi/\varphi_{B2} \rfloor)\xi_B \mu_2 \varphi_{B2} + j\xi_2 \mu_B \varphi_{B2} - k\Theta}{\mu_3} \right\} v_1 \\ &\qquad\qquad + \frac{i\varphi_{B2} + \{k\psi\}_{\varphi_{B2}}}{\mu_B \varphi_{B2}} v_2 + \frac{j\varphi_{B2} + k}{\mu_2 \varphi_{B2}} v_3; \\ &\qquad i = 0, \ldots, \mu_B - 1; \quad j = 0, \ldots, \mu_2 - 1; \quad k = 0, \ldots, \varphi_{B2} - 1. \end{aligned} \tag{129}$$

6.4.3. *Formulas for* $\Sigma_A, \Sigma'_A, \Sigma_B, \Sigma_l, \Sigma_1, \Sigma'_1, \Sigma_2,$ *and* Σ_3. As in Case III, for some of the sums $\Sigma_\bullet$ and $\Sigma'_\bullet$, we will have to distinguish between two cases. Let us put

$$n^* = \frac{\gcd(y_A, m_1)}{\gcd(y_A, m_1, m(h^*))},$$

$$\text{with} \qquad m(h^*) = \frac{\xi_A y_A + m_1}{\mu_A} \in \mathbf{Z}_{>0} \qquad \text{and} \qquad h^* = \frac{\xi_A}{\mu_A} v_0 + \frac{1}{\mu_A} v_1 \in \mathbf{Z}^3,$$

a generating element of the group H_A (if $\mu_A > 1$). Then we have that

$$p^{\frac{\xi_A(w\cdot v_0)+w\cdot v_1}{\mu_A}} \qquad \text{and} \qquad p^{\frac{w\cdot v_0+\xi'_A(w\cdot v_1)}{\mu_A}}$$

both equal one if $n^* \mid n$, while they both differ from one if $n^* \nmid n$. Proceeding as in Paragraphs 4.13.1–4.13.2, we obtain that

$$\Sigma_A = \begin{cases} \mu_A, & \text{if } n^* \mid n; \\ 0, & \text{otherwise}; \end{cases} \qquad \text{and} \tag{130}$$

$$\frac{\Sigma'_A}{\log p} = \begin{cases} \dfrac{(y_A + m_1)(\mu_A - 1)}{2}, & \text{if } n^* \mid n; \\ \dfrac{y_A}{p^{\frac{w\cdot v_0+\xi'_A(w\cdot v_1)}{\mu_A}} - 1} + \dfrac{m_1}{p^{\frac{\xi_A(w\cdot v_0)+w\cdot v_1}{\mu_A}} - 1}, & \text{otherwise}. \end{cases} \tag{131}$$

We continue as in Paragraph 4.13.3. Based on (126) and $p^{w\cdot v_1} = 1$, we find that

$$\Sigma_B = \begin{cases} \dfrac{F_2}{p^{\frac{\xi_B(w\cdot v_1)+w\cdot v_2}{\mu_B}} - 1}, & \text{in any case}; \\ \dfrac{F_2}{q^{\varphi_{AB}} - 1}, & \text{if } n^* \mid n. \end{cases} \tag{132}$$

The special formula for $n^* \mid n$ arises from

$$p^{\frac{w\cdot v_0+\xi'_A(w\cdot v_1)}{\mu_A}} p^{\frac{\xi_B(w\cdot v_1)+w\cdot v_2}{\mu_B}} = p^{\varphi_{AB}(-s_0-1)} = q^{\varphi_{AB}}, \tag{133}$$

which in turn follows from $v_0 + \xi'_A v_1 \in \mu_A \mathbf{Z}^3$, $\xi_B v_1 + v_2 \in \mu_B \mathbf{Z}^3$, (118), and (121). For Σ_l we use (125), $p^{w\cdot v_0} = 1$, and (120) in order to conclude

$$\Sigma_l = \sum_{h\in H_l} p^{w\cdot h} = \sum_{j=0}^{\mu_l - 1} \left(p^{\frac{-b_3(w\cdot v_0)+w\cdot v_3}{\mu_l}}\right)^j = \sum_j q^j = \frac{F_3}{q-1}.$$

By (124), (127), and (117) we have that

$$v_0 + \xi'_A v_1 \in \mu_A \mathbf{Z}^3, \qquad \xi_2 v_1 + v_3 \in \mu_2 \mathbf{Z}^3, \qquad \text{and} \qquad b_3 v_0 - v_3 \in \mu_l \mathbf{Z}^3.$$

Since $\mu_2 = \gcd(\mu_A, \mu_l)$, we obtain

$$-b_3(v_0 + \xi'_A v_1) + (\xi_2 v_1 + v_3) + (b_3 v_0 - v_3) = (-\xi'_A b_3 + \xi_2) v_1 \in \mu_2 \mathbf{Z}^3,$$

and hence

$$\frac{-\xi'_A b_3 + \xi_2}{\mu_2} \in \mathbf{Z}.$$

Using $\psi\mu_2 = \Psi = b_3\mu_A$, $p^{w\cdot v_1} = 1$, (120), and $\mu_l/\mu_2 = \mu_l' \in \mathbf{Z}_{>0}$, it then follows that

$$(134)\quad \left(p^{\frac{w\cdot v_0+\xi_A'(w\cdot v_1)}{\mu_A}}\right)^{-\psi} p^{\frac{\xi_2(w\cdot v_1)+w\cdot v_3}{\mu_2}} = p^{\frac{-\Psi(w\cdot v_0)+(-\xi_A'\Psi+\xi_2\mu_A)(w\cdot v_1)+\mu_A(w\cdot v_3)}{\mu_A\mu_2}}$$
$$= p^{\frac{-b_3(w\cdot v_0)+(-\xi_A' b_3+\xi_2)(w\cdot v_1)+w\cdot v_3}{\mu_2}} = p^{\frac{-b_3(w\cdot v_0)+w\cdot v_3}{\mu_2}} = p^{\frac{\mu_l(-s_0-1)}{\mu_2}} = q^{\mu_l'}.$$

In this way (127) yields
(135)
$$\Sigma_2 = \sum_{h\in H_2} p^{w\cdot h} = \sum_{j=0}^{\mu_2-1}\left(p^{\frac{\xi_2(w\cdot v_1)+w\cdot v_3}{\mu_2}}\right)^j = \begin{cases} \dfrac{F_3}{p^{\frac{\xi_2(w\cdot v_1)+w\cdot v_3}{\mu_2}}-1}, & \text{in any case;} \\ \dfrac{F_3}{q^{\mu_l'}-1}, & \text{if } n^* \mid n. \end{cases}$$

Keeping in mind that $p^{w\cdot v_0} = 1$, we easily find from the description (128) of the points of H_1 that

$$\Sigma_1 = \sum_{h\in H_1} p^{w\cdot h} = \sum_{i=0}^{\mu_A-1}\sum_{j=0}^{\mu_l-1} p^{\frac{i\xi_A\mu_l-jb_3\mu_A}{\mu_1}(w\cdot v_0)+\frac{i}{\mu_A}(w\cdot v_1)+\frac{j}{\mu_l}(w\cdot v_3)}$$
$$= \sum_i \left(p^{\frac{\xi_A(w\cdot v_0)+w\cdot v_1}{\mu_A}}\right)^i \sum_j \left(p^{\frac{-b_3(w\cdot v_0)+w\cdot v_3}{\mu_l}}\right)^j$$
$$(136)\quad = \Sigma_A\Sigma_l = \begin{cases} \dfrac{\mu_A F_3}{q-1}, & \text{if } n^* \mid n; \\ 0, & \text{otherwise.} \end{cases}$$

To calculate Σ_1' we follow the same process as in Case III. Write ψ as $\psi = t\mu_l' + \overline{\psi}$ with $t \in \mathbf{Z}_{\geqslant 0}$ and $\overline{\psi} = \{\psi\}_{\mu_l'}$. Clearly $\overline{\psi} \in \{0, \ldots, \mu_l'-1\}$, and since $\gcd(\psi, \mu_l') = 1$, we also have $\gcd(\overline{\psi}, \mu_l') = 1$. Hence $\overline{\psi} = 0$ occurs if and only if $\mu_l' = 1$. Exclusively in the case that $\mu_l' > 1$ we also introduce the numbers

$$\kappa_\rho = \min\left\{\kappa \in \mathbf{Z}_{\geqslant 0} \;\middle|\; \left\lfloor \frac{\kappa\overline{\psi}}{\mu_l'} \right\rfloor = \rho\right\} = \left\lceil \frac{\rho\mu_l'}{\overline{\psi}} \right\rceil; \qquad \rho = 0, \ldots, \overline{\psi}.$$

Proceeding as in Subsection 4.15 and applying (119) in the end, we eventually obtain that
(137)

$$\frac{\Sigma_1'}{(\log p)F_3} = \begin{cases} \dfrac{1}{1-q}\left(\dfrac{y_A}{q^{\mu_l'}-1}\displaystyle\sum_{\rho=1}^{\overline{\psi}} q^{\kappa_\rho} - \dfrac{(y_A+m_1)(\mu_A-1)}{2} \right. \\ \qquad \left. + \dfrac{y_A t-\mu_A}{1-q^{-1}} - \dfrac{m_3\mu_A(F_3+1)}{F_3} - y_A\right) - \dfrac{y_A}{q^{\mu_l'}-1}, & \text{if } n^*\mid n; \\ \dfrac{y_A p^{\frac{w\cdot v_0+\xi_A'(w\cdot v_1)}{\mu_A}}}{\left(p^{\frac{w\cdot v_0+\xi_A'(w\cdot v_1)}{\mu_A}}-1\right)\left(p^{\frac{\xi_2(w\cdot v_1)+w\cdot v_3}{\mu_2}}-1\right)}. \\ \qquad \displaystyle\sum_{\kappa=0}^{\mu_l'-1} q^{\kappa}\left(p^{\frac{w\cdot v_0+\xi_A'(w\cdot v_1)}{\mu_A}}\right)^{\left\lfloor \frac{\kappa\psi}{\mu_l'}\right\rfloor} \\ \quad - \dfrac{y_A}{p^{\frac{\xi_2(w\cdot v_1)+w\cdot v_3}{\mu_2}}-1} + \dfrac{m_1}{(q-1)\left(p^{\frac{\xi_A(w\cdot v_0)+w\cdot v_1}{\mu_A}}-1\right)}, & \text{if } n^*\nmid n; \end{cases}$$

thereby adopting the convention that the empty sum over ρ equals zero if $\mu_l'=1$.

From the description (129) of the points of H_3, it is reasonable that also the calculation of Σ_3 is essentially not different from the one in Case III; proceeding as in Paragraph 4.17.3, thereby using Identity (122), we find that
(138)

$$\Sigma_3 = \frac{F_2F_3}{\left(p^{\frac{\xi_B(w\cdot v_1)+w\cdot v_2}{\mu_B}}-1\right)\left(p^{\frac{\xi_2(w\cdot v_1)+w\cdot v_3}{\mu_2}}-1\right)} \sum_{k=0}^{\varphi_{B2}-1} q^k\left(p^{\frac{\xi_B(w\cdot v_1)+w\cdot v_2}{\mu_B}}\right)^{-\left\lfloor\frac{k\psi}{\varphi_{B2}}\right\rfloor}.$$

A simplified version of this formula, valid in the case that $n^*\mid n$ and justified by Equalities (133) and (134), is given by

$$\Sigma_3 = \frac{F_2F_3}{(q^{\varphi_{AB}}-1)(q^{\mu_l'}-1)} \sum_{k=0}^{\varphi_{B2}-1} q^{k-\varphi_{AB}\left\lfloor\frac{k\psi}{\varphi_{B2}}\right\rfloor}. \tag{139}$$

6.5. **Proof of $R_2' = R_1' = 0$.** On the one hand, it is clear from (115, 130, and 136) that $R_2'=0$ in any case. On the other hand, if we fill in Formulas (130–132, 135–139) for the $\Sigma(\cdot)(s_0)$ and the $\Sigma(\cdot)'(s_0)$ in Expression (116) for R_1', we see that proving $R_1'=0$ comes down to verifying that

$$\sum_{k=0}^{\varphi_{B2}-1} q^{k-\varphi_{AB}\left\lfloor\frac{k\psi}{\varphi_{B2}}\right\rfloor} = \frac{q^{\varphi_{AB}}-1}{q-1}\sum_{\rho=1}^{\overline{\psi}} q^{\kappa_\rho} + \frac{q^{\mu_l'}-1}{q-1}\left(tq\frac{q^{\varphi_{AB}}-1}{q-1}+1\right),$$

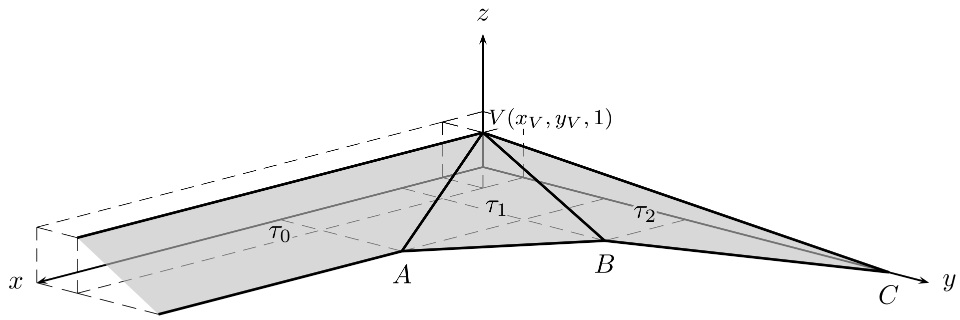


FIGURE 10. Case VI: B_1-facets τ_0, τ_1, and τ_2 all contribute to s_0

if $n^* \mid n$, and

$$\Bigl(p^{\frac{w\cdot v_0+\xi'_A(w\cdot v_1)}{\mu_A}}-1\Bigr)\sum_{k=0}^{\varphi_{B2}-1} q^k\Bigl(p^{\frac{\xi_B(w\cdot v_1)+w\cdot v_2}{\mu_B}}\Bigr)^{-\left\lfloor\frac{k\psi}{\varphi_{B2}}\right\rfloor} + p^{\frac{w\cdot v_0+\xi'_A(w\cdot v_1)}{\mu_A}}\Bigl(p^{\frac{\xi_B(w\cdot v_1)+w\cdot v_2}{\mu_B}}-1\Bigr)\sum_{\kappa=0}^{\mu'_l-1} q^\kappa\Bigl(p^{\frac{w\cdot v_0+\xi'_A(w\cdot v_1)}{\mu_A}}\Bigr)^{\left\lfloor\frac{\kappa\psi}{\mu'_l}\right\rfloor} = \Bigl(p^{\frac{\xi_2(w\cdot v_1)+w\cdot v_3}{\mu_2}}-1\Bigr)\frac{q^{\varphi_{AB}}-1}{q-1},$$

otherwise. In order to obtain this last equation, we need to apply Identity (133) at some point. Since the analogous relations between the variables hold, e.g.,

$$\varphi_{B2}=\varphi_{AB}\psi+\mu'_l \qquad \text{and} \qquad \psi=t\mu'_l+\overline{\psi},$$

these final assertions can be proved in exactly the same way as in Subsection 4.18 of Case III. Hence we conclude Case V.

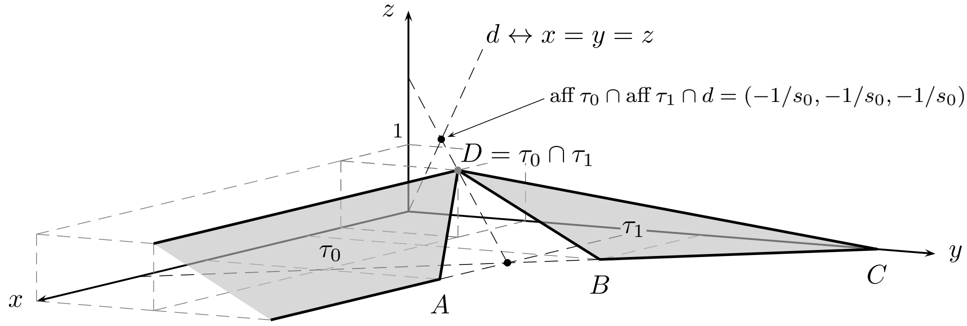
7. Case VI: at least three facets of Γ_f contribute to s_0; all of them are B_1-facets (compact or not) with respect to a same variable and they are 'connected to each other by edges'

More precisely, we mean that we can denote the contributing B_1-facets by $\tau_0, \tau_1, \dots, \tau_t$ with $t \geqslant 2$ in such a way that facets τ_{i-1} and τ_i share an edge for all $i \in \{1, \dots, t\}$. An example with $t = 2$ is shown in Figure 10.

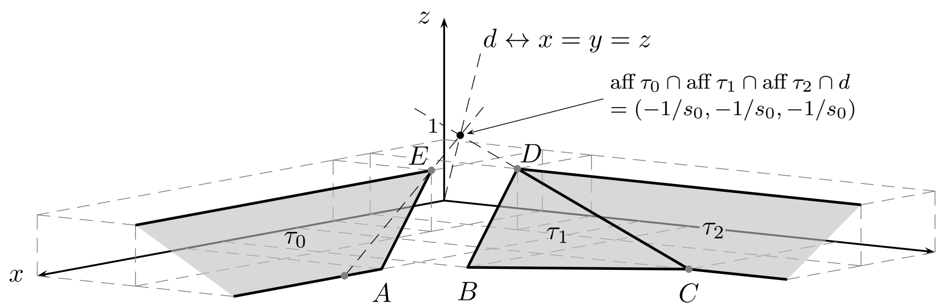
Let us assume that the contributing facets $\tau_0, \tau_1, \dots, \tau_t$ are B_1 with respect to the variable z. Since the τ_i all contribute to the same candidate pole s_0, their affine supports intersect the diagonal of the first octant in the same point $(-1/s_0, -1/s_0, -1/s_0)$. As these affine supports share only one point, the aforementioned intersection point must be the contributing facets' common vertex V at 'height' one:

$$\left(-\frac{1}{s_0}, -\frac{1}{s_0}, -\frac{1}{s_0}\right) = (x_V, y_V, 1).$$

We conclude that $x_V = y_V = 1$ and $s_0 = -1$. Hence under the conditions of Theorem 0.40, Case VI cannot occur.



(a) Non-compact B_1-facet τ_0 and B_1-simplex τ_1 both contribute to s_0. As they have only one point in common, they form two separate clusters.



(b) B_1-Facets τ_0, τ_1, τ_2 all contribute to s_0. We distinguish the clusters $\{\tau_0\}$ and $\{\tau_1, \tau_2\}$.

Figure 11. General Case: several 'clusters' of B_1-facets contribute to the candidate-pole s_0

8. General case: several groups of B_1-facets contribute to s_0; every group is separately covered by one of the previous cases, and the groups have pairwise at most one point in common

As the different 'clusters' of contributing B_1-facets pairwise share not more than one point, we can decompose each cone associated to a vertex of Γ_f into simplicial cones in such a way that the relevant residues in s_0 split up into parts, each part corresponding to one of the preceding cases. In this way the general case follows immediately from the previous ones. Figure 11 shows two possible configurations of B_1-facets that fall under the general case.

9. The main theorem for a non-trivial character of $\mathbf{Z}_p^\times$

In this section we consider Igusa's zeta function of a polynomial $f(x_1, \ldots, x_n) \in \mathbf{Z}_p[x_1, \ldots, x_n]$ and a character $\chi : \mathbf{Z}_p^\times \to \mathbf{C}^\times$ of $\mathbf{Z}_p^\times$, and we prove the analogue of Theorem 0.12 for a non-trivial character. We start with the definition of this 'twisted' p-adic zeta function.

Let p be a prime number and $a \in \mathbf{Q}_p$. We denote the p-adic order of a by $\operatorname{ord}_p a \in \mathbf{Z} \cup \{\infty\}$; we write $|a| = p^{-\operatorname{ord}_p a}$ for the p-adic norm of a and $\operatorname{ac} a = |a|a$

for its angular component. As before, we denote by $|dx| = |dx_1 \wedge \cdots \wedge dx_n|$ the Haar measure on $\mathbf{Q}_p^n$, normalized in such a way that $\mathbf{Z}_p^n$ has measure one.

Definition 9.1 (local twisted Igusa zeta function)**.** Cfr. [24, Def. 1.1]. Let p be a prime number and $f(x) = f(x_1, \ldots, x_n)$ a polynomial in $\mathbf{Z}_p[x_1, \ldots, x_n]$. Let $\chi : \mathbf{Z}_p^\times \to \mathbf{C}^\times$ be a character of $\mathbf{Z}_p^\times$, i.e., a multiplicative group homomorphism with finite image. We formally put $\chi(0) = 0$. To f and χ we associate the local Igusa zeta function

$$Z_{f,\chi}^0 : \{s \in \mathbf{C} \mid \Re(s) > 0\} \to \mathbf{C} : s \mapsto \int_{p\mathbf{Z}_p^n} \chi(\operatorname{ac} f(x))|f(x)|^s |dx|.$$

If χ is the trivial character, we obtain the usual local Igusa zeta function of f. In this section we will deal with the non-trivial characters. The rationality result of Igusa [26] and Denef [11] holds for the above version of Igusa's zeta function as well. From now on, by $Z_{f,\chi}^0$ we mean the meromorphic continuation to $\mathbf{C}$ of the function defined in Definition 9.1.

The goal is to verify the following analogue of Theorem 0.12.

Theorem 9.2 (Monodromy Conjecture for Igusa's zeta function of a non-degenerate surface singularity and a non-trivial character of $\mathbf{Z}_p^\times$)**.** *Let $f(x, y, z) \in \mathbf{Z}[x, y, z]$ be a nonzero polynomial in three variables satisfying $f(0, 0, 0) = 0$, and let $U \subseteq \mathbf{C}^3$ be a neighborhood of the origin. Suppose that f is non-degenerate over $\mathbf{C}$ with respect to all the compact faces of its Newton polyhedron, and let p be a prime number such that f is also non-degenerate over $\mathbf{F}_p$ with respect to the same faces.*[33] *Let $\chi : \mathbf{Z}_p^\times \to \mathbf{C}^\times$ be a non-trivial character of $\mathbf{Z}_p^\times$, and assume that χ is trivial on $1 + p\mathbf{Z}_p$. Suppose that s_0 is a pole of the local Igusa zeta function $Z_{f,\chi}^0$ associated to f and χ. Then $e^{2\pi i \Re(s_0)}$ is an eigenvalue of the local monodromy of f at some point of $f^{-1}(0) \cap U$.*

The reason that we restrict to characters χ that are trivial on $1 + p\mathbf{Z}_p$, is that in this case we have a nice analogue of Denef and Hoornaert's formula (Theorem 0.27) for $Z_{f,\chi}^0$. We give the formula below, but first we introduce a notation that simplifies the statement of the formula.

Notation 9.3. Let p be a prime number and $\chi : \mathbf{Z}_p^\times \to \mathbf{C}^\times$ a character of $\mathbf{Z}_p^\times$. Assume that χ is trivial on the (multiplicative) subgroup $1 + p\mathbf{Z}_p$ of $\mathbf{Z}_p^\times$. We shall identify the quotient group $\mathbf{Z}_p^\times/(1 + p\mathbf{Z}_p)$ with $\mathbf{F}_p^\times$, and we shall denote by $\pi : \mathbf{Z}_p^\times \to \mathbf{F}_p^\times$ the natural surjective homomorphism. Since $1 + p\mathbf{Z}_p \subseteq \ker \chi$, there exists a unique homomorphism $\overline{\chi} : \mathbf{F}_p^\times \to \mathbf{C}^\times$ such that $\chi = \overline{\chi} \circ \pi$. In order for $\overline{\chi}$ to be defined on the whole of $\mathbf{F}_p$, we shall formally put $\overline{\chi}(0) = 0$.

Theorem 9.4. [24, Thm. 3.4]*. Let p be a prime number; let $f(x) = f(x_1, \ldots, x_n)$ be a nonzero polynomial in $\mathbf{Z}_p[x_1, \ldots, x_n]$ satisfying $f(0) = 0$. Suppose that f is non-degenerate over $\mathbf{F}_p$ with respect to all the compact faces of its Newton polyhedron Γ_f. Let $\chi : \mathbf{Z}_p^\times \to \mathbf{C}^\times$ be a non-trivial character of $\mathbf{Z}_p^\times$, and assume that χ is trivial on $1 + p\mathbf{Z}_p$. Then the local Igusa zeta function associated to f and χ is the meromorphic complex function*

$$Z_{f,\chi}^0 : s \mapsto \sum_{\substack{\tau \text{ compact} \\ \text{face of } \Gamma_f}} L_\tau S(\Delta_\tau)(s),$$

[33]By Remark 0.11(i) this is the case for almost all prime numbers p.

with

$$L_\tau = p^{-n} \sum_{x \in (\mathbf{F}_p^\times)^n} \overline{\chi}\big(\overline{f_\tau}(x)\big)$$

and

$$S(\Delta_\tau)(s) = \sum_{k \in \mathbf{Z}^n \cap \Delta_\tau} p^{-\sigma(k) - m(k)s}$$

for every compact face τ of Γ_f. Hereby $f_\tau, \overline{f_\tau}$, and $\overline{\chi}$ are defined as in Notations 0.6, 0.9, and 9.3, respectively; the definitions of $\sigma(k), m(k)$, and Δ_τ can be found in Notation 0.25 and Definitions 0.13 and 0.16, respectively. The sums $S(\Delta_\tau)(s)$ can be calculated in the same way as in Theorem 0.27.

Note that, contrary to the trivial character case, the L_τ do not depend on the variable s. Consequently, $Z^0_{f,\chi}$ for a non-trivial character χ, has 'fewer' candidate poles than Z^0_f.

Corollary 9.5. *Let f and χ be as in Theorem 9.4. Let $\gamma_1, \ldots, \gamma_r$ be all the facets of Γ_f, and let $v_1, \ldots, v_r$ be the unique primitive vectors in $\mathbf{Z}^n_{\geqslant 0} \setminus \{0\}$ that are perpendicular to $\gamma_1, \ldots, \gamma_r$, respectively. From Theorem 9.4 and the rational expression for $S(\Delta_\tau)(s)$ obtained in Theorem 0.27, it follows that the poles of $Z^0_{f,\chi}$ are among the numbers*

$$-\frac{\sigma(v_j)}{m(v_j)} + \frac{2k\pi i}{m(v_j)\log p}, \tag{140}$$

with $j \in \{1, \ldots, r\}$ such that $m(v_j) \neq 0$, and $k \in \mathbf{Z}$. We shall refer to these numbers as the candidate poles of $Z^0_{f,\chi}$.

Now suppose that f, U, p, χ, and s_0 are as in Theorem 9.2. Then s_0 is one of the numbers (140). Theorem 0.34 tells us that if s_0 is contributed (cfr. Definition 0.31) by a facet of Γ_f that is not a B_1-facet (cfr. Definition 0.33), then $e^{2\pi i \Re(s_0)}$ is an eigenvalue of monodromy of f at some point of $f^{-1}(0) \cap U$. Proposition 0.39 says that the same is true if s_0 is contributed by two B_1-facets of Γ_f that are not B_1 for a same variable and that have an edge in common. Therefore, in order to obtain Theorem 9.2, it is sufficient to verify the following proposition.

Proposition 9.6 (On candidate poles of $Z^0_{f,\chi}$ only contributed by B_1-facets)**.** *Let p be a prime number and let $f(x, y, z) \in \mathbf{Z}_p[x, y, z]$ be a nonzero polynomial in three variables with $f(0, 0, 0) = 0$. Suppose that f is non-degenerate over $\mathbf{F}_p$ with respect to all the compact faces of its Newton polyhedron. Let $\chi : \mathbf{Z}_p^\times \to \mathbf{C}^\times$ be a non-trivial character of $\mathbf{Z}_p^\times$ that is trivial on $1 + p\mathbf{Z}_p$. Suppose that s_0 is a candidate pole of $Z^0_{f,\chi}$ that is only contributed by B_1-facets of Γ_f. Further assume that for any pair of contributing B_1-facets, we have that*

- *either they are B_1-facets for a same variable,*
- *or they have at most one point in common.*

Then s_0 is not a pole of $Z^0_{f,\chi}$.

Let p, f, χ, and s_0 be as in the proposition. Let us consider the same seven cases as in the proof of Theorem 0.40. The three observations below show that in every case, the relevant terms in the formula for $Z^0_{f,\chi}$ from Theorem 9.4, are either zero or they cancel in pairs. In what follows we shall use the notations of Theorem 0.27.

First consider a vertex $V(x_V, y_V, 1)$ of Γ_f at 'height' one. The corresponding polynomial $\overline{f_V}$ has the form $\overline{f_V} = \overline{a_V} x^{x_V} y^{y_V} z$ with $\overline{a_V} \in \mathbf{F}_p^\times$. The factor L_V is thus given by

$$\begin{aligned} L_V &= p^{-3} \sum_{(x,y,z)\in(\mathbf{F}_p^\times)^3} \overline{\chi}\big(\overline{a_V} x^{x_V} y^{y_V} z\big) \\ &= p^{-3} \overline{\chi}(\overline{a_V}) \sum_{x\in\mathbf{F}_p^\times} \overline{\chi}^{x_V}(x) \sum_{y\in\mathbf{F}_p^\times} \overline{\chi}^{y_V}(y) \sum_{z\in\mathbf{F}_p^\times} \overline{\chi}(z). \end{aligned} \tag{141}$$

Since χ is non-trivial but trivial on $1+p\mathbf{Z}_p$, the character $\overline{\chi}$ of $\mathbf{F}_p^\times$ is also non-trivial. It is well-known that in this case the last sum of (141) equals zero. Indeed, for any $u \in \mathbf{F}_p^\times$ the map $\mathbf{F}_p^\times \to \mathbf{F}_p^\times : z \mapsto uz$ is a permutation. Consequently,

$$\sum_{z\in\mathbf{F}_p^\times} \overline{\chi}(z) = \sum_{z\in\mathbf{F}_p^\times} \overline{\chi}(uz) = \overline{\chi}(u) \sum_{z\in\mathbf{F}_p^\times} \overline{\chi}(z).$$

As $\overline{\chi}$ is non-trivial, there exists a $u \in \mathbf{F}_p^\times$ with $\overline{\chi}(u) \neq 1$, and for such u the above equation implies that $\sum_{z\in\mathbf{F}_p^\times} \overline{\chi}(z) = 0$. We conclude that $L_V = 0$ and the term associated to V vanishes.

Next let us consider a B_1-simplex τ_0, say for the variable z. Let A and B be the two vertices of τ_0 in the plane $\{z = 0\}$, and let $C(x_C, y_C, 1)$ be the vertex of τ_0 at distance one of this plane. For $L_{[AB]}$ we find

$$L_{[AB]} = p^{-3} \sum_{(x,y,z)\in(\mathbf{F}_p^\times)^3} \overline{\chi}\big(\overline{f_{[AB]}}(x,y)\big) = p^{-3}(p-1) \sum_{(x,y)\in(\mathbf{F}_p^\times)^2} \overline{\chi}\big(\overline{f_{[AB]}}(x,y)\big),$$

while L_{τ_0} is given by

$$L_{\tau_0} = p^{-3} \sum_{(x,y)\in(\mathbf{F}_p^\times)^2} \sum_{z\in\mathbf{F}_p^\times} \overline{\chi}\big(\overline{f_{[AB]}}(x,y) + \overline{a_C} x^{x_C} y^{y_C} z\big) \tag{142}$$

for some $\overline{a_C} \in \mathbf{F}_p^\times$. Fix $(x,y) \in (\mathbf{F}_p^\times)^2$. If z runs through $\mathbf{F}_p^\times$, the argument of $\overline{\chi}$ in (142) runs through all elements of the set $\mathbf{F}_p \setminus \big\{\overline{f_{[AB]}}(x,y)\big\}$. Consequently,

$$\begin{aligned} L_{\tau_0} &= p^{-3} \sum_{(x,y)\in(\mathbf{F}_p^\times)^2} \Big(\textstyle\sum_{u\in\mathbf{F}_p} \overline{\chi}(u) - \overline{\chi}\big(\overline{f_{[AB]}}(x,y)\big)\Big) \\ &= -p^{-3} \sum_{(x,y)\in(\mathbf{F}_p^\times)^2} \overline{\chi}\big(\overline{f_{[AB]}}(x,y)\big). \end{aligned}$$

Together with the fact that

$$S(\Delta_{\tau_0})(s) = \frac{1}{p^{\sigma_0+m_0 s} - 1} \qquad \text{and} \qquad S(\Delta_{[AB]})(s) = \frac{1}{(p^{\sigma_0+m_0 s} - 1)(p-1)}$$

(with (m_0, σ_0) the numerical data associated to τ_0), we now easily find that the sum of the terms associated to τ_0 and $[AB]$ equals zero.

Finally, consider any B_1-facet τ_0 (compact or not), and assume that τ_0 is B_1 for the variable z. Let A be a vertex of τ_0 in the plane $\{z = 0\}$ and $C(x_C, y_C, 1)$ the vertex of τ_0 at 'height' one. Denote by τ_1 the other facet of Γ_f that contains the edge $[AC]$, and let τ_2 be the facet in $\{z = 0\}$. Denote by δ_A the simplicial subcone of Δ_A strictly positively spanned by the primitive vectors $v_0, v_1, v_2 \in \mathbf{Z}^3_{\geqslant 0} \setminus \{0\}$

that are perpendicular to τ_0, τ_1, τ_2, respectively. In the same way as in the previous paragraph we find that

$$L_A = -(p-1)L_{[AC]}.$$

If we combine this identity with the expressions

$$S(\Delta_{[AC]})(s) = \frac{\Sigma(\Delta_{[AC]})(s)}{(p^{\sigma_0+m_0 s}-1)(p^{\sigma_1+m_1 s}-1)} \qquad \text{and}$$

$$S(\delta_A)(s) = \frac{\Sigma(\Delta_{[AC]})(s)}{(p^{\sigma_0+m_0 s}-1)(p^{\sigma_1+m_1 s}-1)(p-1)}$$

(where (m_0, σ_0) and (m_1, σ_1) denote the numerical data of τ_0 and τ_1, respectively), we find again that the terms associated to $[AC]$ and δ_A cancel out.

This concludes (the sketch of) the proof of Proposition 9.6 and Theorem 9.2.

10. The main theorem in the motivic setting

10.1. **The local motivic zeta function and the motivic Monodromy Conjecture.** The theory of motivic integration was invented by Kontsevich and further developed by among others Denef–Loeser [17, 18, 19], Loeser–Sebag [33, 39], and Cluckers–Loeser [10]. Denef and Loeser introduced the motivic zeta function and the corresponding Monodromy Conjecture in [16]. For an introduction to motivic integration, motivic zeta functions, and the (motivic) Monodromy Conjecture, we refer to [36] and [44]. In this section we will only give the definitions that are needed to state the results.

In motivic integration theory, one associates to each algebraic variety X over $\mathbf{C}$, and to each $l \in \mathbf{Z}_{\geqslant 0}$, a space $\mathcal{L}_l(X)$ of so-called l-jets on X. Informally speaking, this jet space $\mathcal{L}_l(X)$ is an algebraic variety over $\mathbf{C}$ whose points with coordinates in $\mathbf{C}$ correspond to points of X with coordinates in $\mathbf{C}[t]/(t^{l+1})$, and vice versa. For all $l' \geqslant l$, there are natural *truncation maps* $\pi_l^{l'} : \mathcal{L}_{l'}(X) \to \mathcal{L}_l(X)$, sending l'-jets to their reduction modulo t^{l+1}.

Next one obtains the space $\mathcal{L}(X)$ of *arcs* on X as the inverse limit $\varprojlim \mathcal{L}_l(X)$ of the system $\big((\mathcal{L}_l(X))_{l\geqslant 0}, (\pi_l^{l'})_{l'\geqslant l\geqslant 0}\big)$. The arc space $\mathcal{L}(X)$ should be thought of as an 'algebraic variety of infinite dimension' over $\mathbf{C}$ whose points with coordinates in $\mathbf{C}$ agree with the points of X with coordinates in $\mathbf{C}[[t]]$. It comes together with natural truncation maps $\pi_l : \mathcal{L}(X) \to \mathcal{L}_l(X)$, sending arcs to their reduction modulo t^{l+1}.

In this section, the only algebraic variety we will consider, is the n-dimensional affine space $X = \mathbf{A}^n(\mathbf{C})$. In this case, $\mathcal{L}_l(\mathbf{A}^n(\mathbf{C})) \cong \mathbf{A}^{n(l+1)}(\mathbf{C})$ and $\mathcal{L}(\mathbf{A}^n(\mathbf{C}))$ can be identified with $(\mathbf{C}[t]/(t^{l+1}))^n$ and $(\mathbf{C}[[t]])^n$, respectively. We will use these identifications throughout the section. The truncation maps are as expected:

$$\pi_l^{l'} : (\mathbf{C}[t]/(t^{l'+1}))^n \to (\mathbf{C}[t]/(t^{l+1}))^n : \big(\varphi_\rho + (t^{l'+1})\big)_\rho \mapsto \big(\varphi_\rho + (t^{l+1})\big)_\rho,$$

$$\pi_l : (\mathbf{C}[[t]])^n \to (\mathbf{C}[t]/(t^{l+1}))^n : \big(\textstyle\sum_\kappa \varphi_{\rho,\kappa} t^\kappa\big)_\rho \mapsto \big(\textstyle\sum_{\kappa=0}^{l} \varphi_{\rho,\kappa} t^\kappa + (t^{l+1})\big)_\rho.$$

In motivic integration, the discrete valuation ring $\mathbf{C}[[t]]$ and its uniformizer t play the role that $\mathbf{Z}_p$ and p play in p-adic integration.

The Grothendieck group of (complex) algebraic varieties is the abelian group $K_0(Var_{\mathbf{C}})$ generated by the isomorphism classes $[X]$ of algebraic varieties X, modulo the relations $[X] = [X \setminus Y] + [Y]$ if Y is Zariski-closed in X. The Grothendieck group is turned into a Grothendieck ring by putting $[X] \cdot [Y] = [X \times Y]$ for all algebraic varieties X and Y. The class of a (complex) algebraic variety in the Grothendieck ring is the universal invariant of an algebraic variety with respect to the additive and multiplicative relations above; it is a refinement of the topological Euler characteristic.

We call a subset C of an algebraic variety X constructible if it can be written as a finite disjoint union of locally closed[34] subvarieties $Y_1, \dots, Y_r$ of X. For such a constructible subset $C = \bigsqcup_j Y_j$, the class $[C] = \sum_j [Y_j]$ of C in the Grothendieck ring is well-defined, i.e., is independent of the chosen decomposition. We denote the class of a point by 1 and the class of the affine line $\mathbf{A}^1(\mathbf{C})$ by $\mathbb{L}$. Finally, we denote by $\mathcal{M}_{\mathbf{C}} = K_0(Var_{\mathbf{C}})[\mathbb{L}^{-1}]$ the localization of $K_0(Var_{\mathbf{C}})$ with respect to $\mathbb{L}$. It is known that $K_0(Var_{\mathbf{C}})$ is not a domain [37]; however, it is still an open question whether $\mathcal{M}_{\mathbf{C}}$ is a domain or not.

We shall call a subset A of $(\mathbf{C}[[t]])^n$ cylindric if $A = \pi_l^{-1}(C)$ for some $l \in \mathbf{Z}_{\geqslant 0}$ and some constructible subset C of $(\mathbf{C}[t]/(t^{l+1}))^n$. For such a cylindric subset $A = \pi_l^{-1}(C)$, one has that

$$\pi_{l'}(A) \cong C \times \mathbf{A}^{n(l'-l)}(\mathbf{C}) \qquad \text{for all } l' \geqslant l;$$

therefore,

$$\mu(A) = [C]\mathbb{L}^{-n(l+1)} = \lim_{l' \to \infty} [\pi_{l'}(A)]\mathbb{L}^{-n(l'+1)} \in \mathcal{M}_{\mathbf{C}}$$

is independent of l. We call $\mu(A)$ the naive motivic measure of A. Its definition and in particular the chosen normalization are inspired by the p-adic Haar measure; note that $\mu((t^l\mathbf{C}[[t]])^n) = \mathbb{L}^{-nl}$ for all $l \in \mathbf{Z}_{\geqslant 0}$.

For $\varphi_\rho = \varphi_{\rho,0} + \varphi_{\rho,1}t + \varphi_{\rho,2}t^2 + \cdots \in \mathbf{C}[[t]] \setminus \{0\}$, we define $\operatorname{ord}_t \varphi_\rho$ as the smallest $\kappa \in \mathbf{Z}_{\geqslant 0}$ such that $\varphi_{\rho,\kappa} \neq 0$; additionally, we agree that $\operatorname{ord}_t 0 = \infty$. If $\varphi = (\varphi_1, \dots, \varphi_n) \in (\mathbf{C}[[t]])^n$, then we put

$$\operatorname{ord}_t \varphi = (\operatorname{ord}_t \varphi_1, \dots, \operatorname{ord}_t \varphi_n) \in (\mathbf{Z}_{\geqslant 0} \cup \{\infty\})^n.$$

Let us recall the definition of the local p-adic zeta function. If $f(x) = f(x_1, \dots, x_n)$ is a nonzero polynomial in $\mathbf{Z}_p[x_1, \dots, x_n]$ with $f(0) = 0$, then

$$\begin{aligned} Z_f^0(s) &= \int_{p\mathbf{Z}_p^n} |f(x)|^s |dx| \\ &= \sum_{l \geqslant 1} \mu(\{x \in p\mathbf{Z}_p^n \mid \operatorname{ord}_p f(x) = l\}) p^{-ls} \\ &= p^{-n} \sum_{l \geqslant 1} \#\{x + p^{l+1}\mathbf{Z}_p^n \in \big(p\mathbf{Z}_p/p^{l+1}\mathbf{Z}_p\big)^n \mid \operatorname{ord}_p f(x) = l\} \cdot (p^{-n}p^{-s})^l, \end{aligned}$$

with $\mu(\cdot)$ the Haar measure on $\mathbf{Q}_p^n$, so normalized that $\mu(\mathbf{Z}_p^n) = 1$. This is the motivation for the following definition.

Definition 10.1 (Local motivic zeta function)**.** Let $f(x) = f(x_1, \dots, x_n)$ be a nonzero polynomial in $\mathbf{C}[x_1, \dots, x_n]$ satisfying $f(0) = 0$. Put

$$\mathcal{X}_l^0 = \{\varphi + \big(t^{l+1}\mathbf{C}[t]\big)^n \in \big(t\mathbf{C}[t]/t^{l+1}\mathbf{C}[t]\big)^n \mid \operatorname{ord}_t f(\varphi) = l\}$$

[34] We mean locally closed w.r.t. the Zariski-topology on X.

for $l \in \mathbf{Z}_{>0}$. Then the local motivic zeta function $Z_f^{\mathrm{mot},0}(s)$ associated to f is by definition the following element of $\mathcal{M}_{\mathbf{C}}[[\mathbb{L}^{-s}]]$:

$$\begin{aligned} Z_f^{\mathrm{mot},0}(s) &= \sum_{l \geqslant 1} \mu(\{\varphi \in (t\mathbf{C}[[t]])^n \mid \mathrm{ord}_t\, f(\varphi) = l\})(\mathbb{L}^{-s})^l \\ &= \mathbb{L}^{-n} \sum_{l \geqslant 1} [\mathcal{X}_l^0](\mathbb{L}^{-n}\mathbb{L}^{-s})^l \in \mathcal{M}_{\mathbf{C}}[[\mathbb{L}^{-s}]]. \end{aligned}$$

Here $\mathbb{L}^{-s}$ should be seen as a formal indeterminate. In what follows we shall always denote $\mathbb{L}^{-s}$ by T; i.e., we define the local motivic zeta function $Z_f^{\mathrm{mot},0}(T)$ of f as

$$Z_f^{\mathrm{mot},0}(T) = \mathbb{L}^{-n} \sum_{l \geqslant 1} [\mathcal{X}_l^0](\mathbb{L}^{-n}T)^l \in \mathcal{M}_{\mathbf{C}}[[T]].$$

The (local) motivic zeta function $Z_f^{\mathrm{mot},0}(T)$ is thus by definition a formal power series in T with coefficients in $\mathcal{M}_{\mathbf{C}}$. By means of resolutions of singularities, Denef and Loeser proved that it is also a rational function in T. More precisely, they proved that there exists a finite set $S \subseteq \mathbf{Z}_{>0}^2$ such that

$$Z_f^{\mathrm{mot},0}(T) \in \mathcal{M}_{\mathbf{C}} \left[\frac{\mathbb{L}^{-\sigma}T^m}{1 - \mathbb{L}^{-\sigma}T^m} \right]_{(m,\sigma) \in S} \subseteq \mathcal{M}_{\mathbf{C}}[[T]].$$

Denef and Loeser also formulated a motivic version of the Monodromy Conjecture. One should be careful, however, when translating the p-adic (or topological) statement of the conjecture to the motivic setting; since it is not known whether $\mathcal{M}_{\mathbf{C}}$ is a domain or not, the notion of pole of $Z_f^{\mathrm{mot},0}(T)$ is not straightforward.

Conjecture 10.2 (Motivic Monodromy Conjecture)**.** *Let $f(x) = f(x_1, \ldots, x_n)$ be a nonzero polynomial in $\mathbf{C}[x_1, \ldots, x_n]$ satisfying $f(0) = 0$. Then there exists a finite set $S \subseteq \mathbf{Z}_{>0}^2$ such that*

$$Z_f^{\mathrm{mot},0}(T) \in \mathcal{M}_{\mathbf{C}}[T] \left[\frac{1}{1 - \mathbb{L}^{-\sigma}T^m} \right]_{(m,\sigma) \in S} \subseteq \mathcal{M}_{\mathbf{C}}[[T]],$$

and such that, for each $(m, \sigma) \in S$, the complex number $e^{-2\pi i \sigma/m}$ is an eigenvalue of the local monodromy of f at some point of the complex zero locus $f^{-1}(0) \subseteq \mathbf{C}^n$ close to the origin.

The goal of this section is to prove the motivic Monodromy Conjecture for a polynomial in three variables that is non-degenerate over $\mathbf{C}$ with respect to its Newton polyhedron, i.e., to prove the following motivic version of Theorem 0.12.

Theorem 10.3 (Monodromy Conjecture for the local motivic zeta function of a non-degenerate surface singularity)**.** *Let $f(x, y, z) \in \mathbf{C}[x, y, z]$ be a nonzero polynomial in three variables satisfying $f(0, 0, 0) = 0$, and let $U \subseteq \mathbf{C}^3$ be a neighborhood of the origin. Suppose that f is non-degenerate over $\mathbf{C}$ with respect to all the compact faces of its Newton polyhedron. Then there exists a finite set $S \subseteq \mathbf{Z}_{>0}^2$ such that*

$$Z_f^{\mathrm{mot},0}(T) \in \mathcal{M}_{\mathbf{C}}[T] \left[\frac{1}{1 - \mathbb{L}^{-\sigma}T^m} \right]_{(m,\sigma) \in S},$$

and such that, for each $(m, \sigma) \in S$, the complex number $e^{-2\pi i \sigma/m}$ is an eigenvalue of the local monodromy of f at some point of $f^{-1}(0) \cap U \subseteq \mathbf{C}^3$.

We discuss a proof of Theorem 10.3 in Subsection 10.3. The essential formula for this proof is treated in the next subsection.

10.2. **A formula for the local motivic zeta function of a non-degenerate polynomial.** We will prove a combinatorial formula à la Denef–Hoornaert [14] for the local motivic zeta function associated to a polynomial that is non-degenerate over the complex numbers. This was also done (in less detail) by Artal et al. [4] and Guibert [23]. We state the formula below, but first we recall the precise notion of non-degeneracy we will be dealing with.

Definition 10.4 (Non-degenerate over $\mathbf{C}$)**.** Let $f(x) = f(x_1, \ldots, x_n)$ be a nonzero polynomial in $\mathbf{C}[x_1, \ldots, x_n]$ satisfying $f(0) = 0$. We say that f is non-degenerate over $\mathbf{C}$ with respect to all the compact faces of its Newton polyhedron Γ_f, if for every compact face τ of Γ_f, the zero locus $f_\tau^{-1}(0) \subseteq \mathbf{C}^n$ of f_τ has no singularities in $(\mathbf{C}^\times)^n$ (cfr. Notation 0.6).

Looking for an analogue for the motivic zeta function of Denef and Hoornaert's formula for Igusa's p-adic zeta function, one roughly expects to recover their formula with p, p^{-s}, and N_τ replaced by $\mathbb{L}$, T, and the class of $\{x \in (\mathbf{C}^\times)^n \mid f_\tau(x) = 0\}$ in the Grothendieck ring of complex varieties, respectively. We have to be careful however. Neither T^{-1}, nor $(1-\mathbb{L}^{-1})^{-1}$ are elements in $\mathcal{M}_{\mathbf{C}}[[T]]$; especially, whereas $\sum_{\lambda=0}^\infty p^{-\lambda} = (1-p^{-1})^{-1}$ in $\mathbf{R}$, the corresponding $\sum_{\lambda=0}^\infty \mathbb{L}^{-\lambda} = (1-\mathbb{L}^{-1})^{-1}$ does not make sense in $\mathcal{M}_{\mathbf{C}}[[T]]$. To avoid the appearance of T^{-1} in the formula, we adopt a slightly different notion of fundamental parallelepiped; to avoid dividing by $1-\mathbb{L}^{-1}$, we have to treat compact faces lying in coordinate hyperplanes differently.[35]

Theorem 10.5. *Let $f(x) = f(x_1, \ldots, x_n)$ be a nonzero polynomial in $\mathbf{C}[x_1, \ldots, x_n]$ satisfying $f(0) = 0$. Suppose that f is non-degenerate over $\mathbf{C}$ with respect to all the compact faces of its Newton polyhedron Γ_f. Then the local motivic zeta function associated to f is given by*

$$Z_f^{\mathrm{mot},0}(T) = \sum_{\substack{\tau \text{ compact face of } \Gamma_f, \\ \tau \not\subseteq \{x_\rho = 0\} \text{ for all } \rho}} L_\tau S(\Delta_\tau) + \sum_{\substack{\tau \text{ compact face of } \Gamma_f, \\ \tau \subseteq \{x_\rho = 0\} \text{ for some } \rho}} L'_\tau S(\Delta_\tau)' \in \mathcal{M}_{\mathbf{C}}[[T]],$$

where the $L_\tau, S(\Delta_\tau), L'_\tau, S(\Delta_\tau)'$ are as defined below.

For τ not contained in any coordinate hyperplane, we have

$$L_\tau = \left(1 - \mathbb{L}^{-1}\right)^n - \mathbb{L}^{-n}[\mathcal{X}_\tau] \frac{1-T}{1-\mathbb{L}^{-1}T} \in \mathcal{M}_{\mathbf{C}}[[T]],$$

with

$$\mathcal{X}_\tau = \left\{x \in (\mathbf{C}^\times)^n \;\middle|\; f_\tau(x) = 0\right\},$$

and

$$S(\Delta_\tau) = \sum_{k \in \mathbf{Z}^n \cap \Delta_\tau} \mathbb{L}^{-\sigma(k)} T^{m(k)} \in \mathcal{M}_{\mathbf{C}}[[T]].$$

By $S(\Delta_\tau) \in \mathcal{M}_{\mathbf{C}}[[T]]$ we mean, more precisely, the following. First choose a decomposition $\{\delta_i\}_{i \in I}$ of the cone Δ_τ into simplicial cones δ_i without introducing new rays, and put $S(\Delta_\tau) = \sum_{i \in I} S(\delta_i)$, with

$$S(\delta_i) = \sum_{k \in \mathbf{Z}^n \cap \delta_i} \mathbb{L}^{-\sigma(k)} T^{m(k)} \in \mathcal{M}_{\mathbf{C}}[[T]]$$

[35] The reason is that for such a face τ, at least one v among the primitive vectors spanning Δ_τ, has numerical data $(m(v), \sigma(v)) = (0, 1)$.

for all $i \in I$. Then assuming that the cone δ_i is strictly positively spanned by the linearly independent primitive vectors v_j, $j \in J_i$, in $\mathbf{Z}^n_{\geqslant 0} \setminus \{0\}$, the element $S(\delta_i) \in \mathcal{M}_{\mathbf{C}}[[T]]$ is defined as[36]

$$S(\delta_i) = \frac{\tilde{\Sigma}(\delta_i)}{\prod_{j\in J_i}\bigl(1 - \mathbb{L}^{-\sigma(v_j)}T^{m(v_j)}\bigr)} \in \mathcal{M}_{\mathbf{C}}[[T]],$$

with

$$\tilde{\Sigma}(\delta_i) = \sum_h \mathbb{L}^{-\sigma(h)}T^{m(h)} \in \mathcal{M}_{\mathbf{C}}[T],$$

where h runs through the elements of the set

$$\tilde{H}(v_j)_{j\in J_i} = \mathbf{Z}^n \cap \tilde{\Diamond}(v_j)_{j\in J_i},$$

with

$$\tilde{\Diamond}(v_j)_{j\in J_i} = \Bigl\{ \sum\nolimits_{j\in J_i} h_j v_j \Bigm| h_j \in (0,1] \text{ for all } j \in J_i \Bigr\}$$

the fundamental parallelepiped[37] *spanned by the vectors v_j, $j \in J_i$.*

Suppose now that the compact face τ of Γ_f is contained in at least one coordinate hyperplane. Define $P_\tau \subseteq \{1,\ldots,n\}$ such that $\rho \in P_\tau$ if and only if $\tau \subseteq \{(x_1,\ldots,x_n) \in \mathbf{R}^n_{\geqslant 0} \mid x_\rho = 0\}$. Note that $1 \leqslant |P_\tau| \leqslant n-1$ and that f_τ only depends on the variables x_ρ, $\rho \notin P_\tau$. If we put

$$\mathcal{X}'_\tau = \Bigl\{ (x_\rho)_{\rho\notin P_\tau} \in (\mathbf{C}^\times)^{n-|P_\tau|} \Bigm| f_\tau(x_\rho)_{\rho\notin P_\tau} = 0 \Bigr\},$$

then we have

$$L'_\tau = \bigl(1-\mathbb{L}^{-1}\bigr)^{n-|P_\tau|} - \mathbb{L}^{-(n-|P_\tau|)}[\mathcal{X}'_\tau]\frac{1-T}{1-\mathbb{L}^{-1}T} \in \mathcal{M}_{\mathbf{C}}[[T]].$$

Denoting the standard basis of $\mathbf{R}^n$ by $(e_\rho)_{1\leqslant\rho\leqslant n}$, it follows that Δ_τ is strictly positively spanned by the vectors e_ρ, $\rho \in P_\tau$, and one or more other primitive vectors v_j, $j \in J_\tau$, in $\mathbf{Z}^n_{\geqslant 0} \setminus \{0\}$.[38] *Choose a decomposition $\{\delta_i\}_{i\in I}$ of the cone Δ_τ into simplicial cones δ_i without introducing new rays, and assume that δ_i is strictly positively spanned by the linearly independent primitive vectors e_ρ, v_j; $\rho \in P_i, j \in J_i$; with $\emptyset \subseteq P_i \subseteq P_\tau$ and $\emptyset \subsetneq J_i \subseteq J_\tau$. For $i \in I$, put*

$$\begin{aligned} \delta'_i &= \tilde{\Diamond}(e_\rho)_{\rho\in P_i} + \operatorname{cone}(v_j)_{j\in J_i} \\ &= \Bigl\{ \sum\nolimits_{\rho\in P_i} h_\rho e_\rho + \sum\nolimits_{j\in J_i} \lambda_j v_j \Bigm| h_\rho \in (0,1], \lambda_j \in \mathbf{R}_{>0} \text{ for all } \rho, j \Bigr\} \subseteq \delta_i. \end{aligned}$$

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[36]Since τ is not contained in any coordinate hyperplane, all $m(v_j)$ are positive integers. Hence $\Bigl(1 - \mathbb{L}^{-\sigma(v_j)}T^{m(v_j)}\Bigr)^{-1} = \sum_{\lambda=0}^{\infty}\Bigl(\mathbb{L}^{-\sigma(v_j)}T^{m(v_j)}\Bigr)^{\lambda} \in \mathcal{M}_{\mathbf{C}}[[T]]$ for all $j \in \bigcup_{i\in I} J_i$.

[37]This is the fundamental parallelepiped with opposite boundaries as before.

[38]Indeed, as τ is compact and contained in $\bigcap_{\rho\in P_\tau}\{x_\rho = 0\}$, we have that $\dim\tau \leqslant n - |P_\tau| - 1$; hence $\dim\Delta_\tau \geqslant |P_\tau| + 1$.

Then $S(\Delta_\tau)'$ is given by[39]

$$S(\Delta_\tau)' = \sum_{i\in I}\bigl(1-\mathbb{L}^{-1}\bigr)^{|P_\tau|-|P_i|}\sum_{k\in\mathbf{Z}^n\cap\delta_i'}\mathbb{L}^{-\sigma(k)}T^{m(k)}$$
$$= \sum_{i\in I}\bigl(1-\mathbb{L}^{-1}\bigr)^{|P_\tau|-|P_i|}\frac{\tilde{\Sigma}(\delta_i)}{\prod_{j\in J_i}\bigl(1-\mathbb{L}^{-\sigma(v_j)}T^{m(v_j)}\bigr)} \in \mathcal{M}_{\mathbf{C}}[[T]],$$

with

$$\tilde{\Sigma}(\delta_i) = \sum_h \mathbb{L}^{-\sigma(h)}T^{m(h)} \in \mathcal{M}_{\mathbf{C}}[T],$$

where h runs through the elements of the set

$$\tilde{H}(e_\rho, v_j)_{\rho,j} = \mathbf{Z}^n \cap \left\{\sum\nolimits_{\rho\in P_i} h_\rho e_\rho + \sum\nolimits_{j\in J_i} h_j v_j \;\middle|\; h_\rho, h_j \in (0,1] \textit{ for all } \rho, j\right\}.$$

The formula as stated above is obtained from Denef and Hoornaert's formula by first replacing p, p^{-s}, and N_τ by their proper analogues and then rewriting the formula in such a way that everything lives in $\mathcal{M}_{\mathbf{C}}[[T]]$. The proof is naturally similar to its p-adic counterpart, but we have to make adaptations due to some restrictions in comparison with the p-adic case.

One important restriction is that the (naive) motivic measure is not σ-additive. As mentioned earlier, we can no longer give meaning to countable sums of measures as $\sum_{\lambda=0}^{\infty}\mathbb{L}^{-\lambda}$ in $\mathcal{M}_{\mathbf{C}}$. This results in a necessary different treatment of compact faces that are contained in coordinate hyperplanes. It also makes that we have—in some sense—less measurable subsets and therefore less freedom in the way we calculate things. For example, where in the p-adic case we start the proof by splitting up the integration domain $p\mathbf{Z}_p^n$ according to the p-order of its elements, we cannot copy this approach in the present setting, as it would give rise to unmeasurable sets. Another example is the following.

In the p-adic case, when calculating $\int_{p\mathbf{Z}_p^n}|f(x)|^s|dx|$, we could ignore the $x\in p\mathbf{Z}_p^n$ with one or more coordinates equal to zero, because this part of the integration domain has measure zero. In the motivic setting, working with the naive motivic measure, we don't have this luxury; the corresponding $\{(\varphi_1,\dots,\varphi_n)\in(t\mathbf{C}[[t]])^n \mid \varphi_\rho = 0 \text{ for some } \rho\}$ is not a cylindric subset of $(t\mathbf{C}[[t]])^n$, hence is not measurable. In what follows, we adapt some familiar notions to better describe this new situation.

We consider the extended non-negative real numbers $\overline{\mathbf{R}}_{\geqslant 0} = \mathbf{R}_{\geqslant 0}\cup\{\infty\}$ with the usual order '$\leqslant$' and addition '$+$'. We extend the usual multiplication in $\mathbf{R}_{\geqslant 0}$ to a multiplication in $\overline{\mathbf{R}}_{\geqslant 0}$ by putting $\infty\cdot 0 = 0\cdot\infty = 0$ and $\infty\cdot x = x\cdot\infty = \infty$ for $x\in\overline{\mathbf{R}}_{>0} = \mathbf{R}_{>0}\cup\{\infty\}$. This allows us to also extend the dot product on $\mathbf{R}^n_{\geqslant 0}$ to a dot product

$$\cdot : \overline{\mathbf{R}}^n_{\geqslant 0}\times\overline{\mathbf{R}}^n_{\geqslant 0}\to\overline{\mathbf{R}}_{\geqslant 0} : \bigl((x_\rho)_\rho,(y_\rho)_\rho\bigr)\mapsto (x_\rho)_\rho\cdot(y_\rho)_\rho = \sum\nolimits_{\rho=1}^{n} x_\rho y_\rho$$

on $\overline{\mathbf{R}}^n_{\geqslant 0}$. The motivation for this definition is that, in this way,

$$\operatorname{ord}_t\varphi^\omega = \operatorname{ord}_t\varphi_1^{\omega_1}\cdots\varphi_n^{\omega_n} = (\operatorname{ord}_t\varphi_1,\dots,\operatorname{ord}_t\varphi_n)\cdot(\omega_1,\dots,\omega_n) = (\operatorname{ord}_t\varphi)\cdot\omega$$

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[39] Note again that all $m(v_j)$ are positive; therefore, $\Bigl(1-\mathbb{L}^{-\sigma(v_j)}T^{m(v_j)}\Bigr)^{-1}\in\mathcal{M}_{\mathbf{C}}[[T]]$ for all $j\in J_\tau = \bigcup_{i\in I} J_i$.

for $\varphi = (\varphi_1, \ldots, \varphi_n) \in (t\mathbf{C}[[t]])^n$ and $\omega = (\omega_1, \ldots, \omega_n) \in \mathbf{R}^n_{\geqslant 0}$, even if $\varphi_\rho = 0$ for some ρ.

Next we extend $m(\cdot)$ and $F(\cdot)$ to $\overline{\mathbf{R}}^n_{\geqslant 0}$ in the expected way:

$$m(k) = \inf_{x \in \Gamma_f} k \cdot x = \min_{\omega \in \operatorname{supp}(f)} k \cdot \omega \in \overline{\mathbf{R}}_{\geqslant 0}, \qquad F(k) = \{x \in \Gamma_f \mid k \cdot x = m(k)\}$$

for $k \in \overline{\mathbf{R}}^n_{\geqslant 0}$. We have the following properties.

Proposition 10.6. *Let $k \in \overline{\mathbf{R}}^n_{\geqslant 0}$ and put $P_k = \{\rho \mid k_\rho = \infty\} \subseteq \{1, \ldots, n\}$.*

(i) *If $k = (0, \ldots, 0)$ or $m(k) = \infty$, then $F(k) = \Gamma_f$, otherwise $F(k)$ is a proper face of Γ_f;*
(ii) *the face $F(k)$ is compact if and only if $k \in \overline{\mathbf{R}}^n_{>0}$ and $m(k) < \infty$;*
(iii) *if $P_k \neq \emptyset$ and $m(k) < \infty$, then $F(k)$ is contained in $\bigcap_{\rho \in P_k} \{x_\rho = 0\}$.*

The map $F : \overline{\mathbf{R}}^n_{\geqslant 0} \to \{\text{faces of } \Gamma_f\}$ induces an equivalence relation on $\overline{\mathbf{R}}^n_{\geqslant 0}$. For every face τ of Γ_f, we put $\Delta^\infty_\tau = F^{-1}(\tau)$ and call it the (extended) cone associated to τ. These equivalence classes are subject to the following properties.

Proposition 10.7. *Let τ be a face of Γ_f and put $\emptyset \subseteq P_\tau = \{\rho \mid \tau \subseteq \{x_\rho = 0\}\} \subsetneq \{1, \ldots, n\}$. Suppose that Δ_τ is strictly positively spanned by the primitive vectors e_ρ, v_j; $\rho \in P_\tau, j \in J_\tau$; in $\mathbf{Z}^n_{\geqslant 0} \setminus \{0\}$.*[40] *Then we have*

(i) *$\Delta_\tau = \Delta^\infty_\tau \cap \mathbf{R}^n_{\geqslant 0}$;*
(ii) *if $\tau = \Gamma_f$, then $\Delta^\infty_\tau = \{(0, \ldots, 0)\} \cup \{k \in \overline{\mathbf{R}}^n_{\geqslant 0} \mid m(k) = \infty\}$;*
(iii) *if τ is a proper face of Γ_f, then*
$$\Delta^\infty_\tau = \Big\{ \sum\nolimits_{\rho \in P_\tau} \bar{\lambda}_\rho e_\rho + \sum\nolimits_{j \in J_\tau} \lambda_j v_j \;\Big|\; \bar{\lambda}_\rho \in \overline{\mathbf{R}}_{>0}, \lambda_j \in \mathbf{R}_{>0} \text{ for all } \rho, j \Big\};$$
(iv) *in particular, if τ is a proper face not contained in any coordinate hyperplane, then $\Delta^\infty_\tau = \Delta_\tau$.*

Furthermore,

(v) *the family $\{\Delta^\infty_\tau \mid \tau \text{ is a face of } \Gamma_f\}$ of all extended cones forms a partition of $\overline{\mathbf{R}}^n_{\geqslant 0}$, while*
(vi) *$\{\Delta^\infty_\tau \mid \tau \text{ is a compact face of } \Gamma_f\}$ partitions $\{k \in \overline{\mathbf{R}}^n_{>0} \mid m(k) < \infty\}$.*

Let us do some more preliminary work to facilitate the actual proof of the theorem. In the lemmas and corollaries that follow we calculate the (naive) motivic measure of some cylindric subsets of $(t\mathbf{C}[[t]])^n$, but first we introduce a notation.

Notation 10.8. For $K \subseteq \overline{\mathbf{Z}}^n_{>0} = (\mathbf{Z}_{>0} \cup \{\infty\})^n$ and $l \in \mathbf{Z}_{>0}$, we put

$$X_{K,l} = \{\varphi \in (t\mathbf{C}[[t]])^n \mid \operatorname{ord}_t \varphi \in K \text{ and } \operatorname{ord}_t f(\varphi) = l\}.$$

If $k \in \mathbf{Z}^n_{>0}$, then we usually write $X_{k,l}$ instead of $X_{\{k\},l}$.

Lemma 10.9. *Let f be as in Theorem 10.5. Suppose that τ is a compact face of Γ_f, and put $\mathcal{X}_\tau = \{x \in (\mathbf{C}^\times)^n \mid f_\tau(x) = 0\}$. Let $k \in \mathbf{Z}^n \cap \Delta_\tau$ and $l \in \mathbf{Z}_{>0}$. Then $k \in \mathbf{Z}^n_{>0}$, $m(k) \in \mathbf{Z}_{>0}$, and*

$$\mu(X_{k,l}) = \begin{cases} 0, & \text{if } l < m(k); \\ \big((\mathbb{L} - 1)^n - [\mathcal{X}_\tau]\big)\mathbb{L}^{-n-\sigma(k)}, & \text{if } l = m(k); \\ [\mathcal{X}_\tau](\mathbb{L} - 1)\mathbb{L}^{-n+m(k)-\sigma(k)-l}, & \text{if } l > m(k). \end{cases}$$

[40] We agree that $J_\tau = \emptyset$ if $\tau = \Gamma_f$; this is, $\Delta_{\Gamma_f} = \{(0, \ldots, 0)\}$ is strictly positively spanned by the empty set.

Proof. Let $\varphi = (\varphi_1, \dots, \varphi_n) \in (t\mathbf{C}[[t]])^n$ with $\operatorname{ord}_t \varphi = k = (k_1, \dots, k_n)$, and let $\psi = (\psi_1, \dots, \psi_n) \in (\mathbf{C}[[t]]^\times)^n$ be such that $\varphi_\rho = t^{k_\rho}\psi_\rho$ for all ρ. Then, for $\omega = (\omega_1, \dots, \omega_n) \in \mathbf{Z}^n_{\geqslant 0}$, we have that

$$\varphi^\omega = \varphi_1^{\omega_1} \cdots \varphi_n^{\omega_n} = t^{k_1\omega_1}\psi_1^{\omega_1} \cdots t^{k_n\omega_n}\psi_n^{\omega_n} = t^{k\cdot\omega}\psi^\omega.$$

Write

$$f(x) = \sum_{\omega \in \mathbf{Z}^n_{\geqslant 0}} a_\omega x^\omega \qquad \text{and} \qquad f_\tau(x) = \sum_{\omega \in \mathbf{Z}^n \cap \tau} a_\omega x^\omega.$$

It follows from $k \in \Delta_\tau$ that $k \cdot \omega = m(k)$ for all $\omega \in \operatorname{supp}(f) \cap \tau$,[41] whereas $k \cdot \omega \geqslant m(k) + 1$ for $\omega \in \operatorname{supp}(f) \setminus \tau$. Hence we can write $f(\varphi)$ as

$$f(\varphi) = t^{m(k)}\big(f_\tau(\psi) + t\tilde{f}_{\tau,k}(t,\psi)\big),$$

with

$$\tilde{f}_{\tau,k}(t,\psi) = \sum_{\omega \in \operatorname{supp}(f)\setminus\tau} a_\omega t^{k\cdot\omega - m(k) - 1}\psi^\omega.$$

First of all, we see that $\operatorname{ord}_t f(\varphi) \geqslant m(k)$; hence $\mu(X_{k,l}) = 0$ for $l < m(k)$. Secondly, we observe that $\operatorname{ord}_t f(\varphi) = m(k)$ if and only if $\operatorname{ord}_t f_\tau(\psi) = 0$. If we write $\psi = (\psi_{\rho,0} + \psi_{\rho,1}t + \psi_{\rho,2}t^2 + \cdots)_{1\leqslant\rho\leqslant n}$, then $f_\tau(\psi) \in f_\tau(\psi_{1,0}, \dots, \psi_{n,0}) + t\mathbf{C}[[t]]$. Consequently, the set

$$\begin{aligned}\widetilde{X}_{k,m(k)} &= \{\psi \in (\mathbf{C}[[t]]^\times)^n \mid \operatorname{ord}_t f(\varphi) = m(k)\}\\ &= \{\psi \in (\mathbf{C}[[t]]^\times)^n \mid f_\tau(\psi_{1,0}, \dots, \psi_{n,0}) \neq 0\}\\ &= \pi_0^{-1}\big(\pi_0\big(\widetilde{X}_{k,m(k)}\big)\big)\end{aligned}$$

is a cylindric subset of $(\mathbf{C}[[t]])^n$ of motivic measure

$$\mu\big(\widetilde{X}_{k,m(k)}\big) = \big[\pi_0\big(\widetilde{X}_{k,m(k)}\big)\big]\mathbb{L}^{-n} = [(\mathbf{C}^\times)^n \setminus \mathcal{X}_\tau]\mathbb{L}^{-n} = \big((\mathbb{L}-1)^n - [\mathcal{X}_\tau]\big)\mathbb{L}^{-n}.$$

The corresponding set $X_{k,m(k)}$ has motivic measure

$$\mu\big(X_{k,m(k)}\big) = \mathbb{L}^{-\sigma(k)}\mu\big(\widetilde{X}_{k,m(k)}\big) = \big((\mathbb{L}-1)^n - [\mathcal{X}_\tau]\big)\mathbb{L}^{-n-\sigma(k)}.$$

Suppose now that $l > m(k)$. Let us first calculate the measure of

$$X_{k,\geqslant l} = \{\varphi \in (t\mathbf{C}[[t]])^n \mid \operatorname{ord}_t \varphi = k \text{ and } \operatorname{ord}_t f(\varphi) \geqslant l\}.$$

From our expression for $f(\varphi)$ we see that $\operatorname{ord}_t f(\varphi) \geqslant l$ if and only if

$$\operatorname{ord}_t\big(f_\tau(\psi) + t\tilde{f}_{\tau,k}(t,\psi)\big) \geqslant l' = l - m(k) \geqslant 1,$$

or, equivalently, if and only if

$$f_\tau(\psi) + t\tilde{f}_{\tau,k}(t,\psi) \equiv 0 \mod t^{l'}\mathbf{C}[[t]]. \tag{143}$$

Whether ψ satisfies the above condition, only depends on the complex numbers

$$\psi_{\rho,\kappa}; \qquad \rho = 1, \dots, n;\ \kappa = 0, \dots, l'-1.$$

Clearly, for ψ to satisfy (143), it is necessary that $f_\tau(\psi_{1,0}, \dots, \psi_{n,0}) = 0$. Fix such an n-tuple $(\psi_{1,0}, \dots, \psi_{n,0})$. Since f is non-degenerate over $\mathbf{C}$ with respect to τ, there exists a ρ_0 such that $(\partial f_\tau/\partial x_{\rho_0})(\psi_{1,0}, \dots, \psi_{n,0}) \neq 0$. Therefore, Hensel's lifting lemma returns, for every free choice of $(n-1)(l'-1)$ complex numbers

$$\psi_{\rho,\kappa}; \qquad \rho = 1, \dots, \rho_0 - 1, \rho_0 + 1, \dots, n;\ \kappa = 1, \dots, l'-1;$$

[41] Recall that $\operatorname{supp}(f) = \{\omega \in \mathbf{Z}^n_{\geqslant 0} \mid a_\omega \neq 0\}$.

unique $\psi_{\rho_0,1},\ldots,\psi_{\rho_0,l'-1}\in\mathbf{C}$ such that ψ satisfies (143). It follows that

$$\begin{aligned}\widetilde{X}_{k,\geqslant l} &= \{\psi\in(\mathbf{C}[[t]]^\times)^n \mid \operatorname{ord}_t f(\varphi)\geqslant l\}\\ &= \{\psi\in(\mathbf{C}[[t]]^\times)^n \mid f_\tau(\psi)+t\tilde{f}_{\tau,k}(t,\psi)\equiv 0 \bmod t^{l'}\mathbf{C}[[t]]\}\\ &= \pi_{l'-1}^{-1}\bigl(\pi_{l'-1}\bigl(\widetilde{X}_{k,\geqslant l}\bigr)\bigr)\end{aligned}$$

is a cylindric subset of $(\mathbf{C}[[t]])^n$ of motivic measure

$$\begin{aligned}\mu\bigl(\widetilde{X}_{k,\geqslant l}\bigr) &= \bigl[\pi_{l'-1}\bigl(\widetilde{X}_{k,\geqslant l}\bigr)\bigr]\mathbb{L}^{-nl'}\\ &= \bigl[\mathcal{X}_\tau\times\mathbf{C}^{(n-1)(l'-1)}\bigr]\mathbb{L}^{-nl'} = [\mathcal{X}_\tau]\mathbb{L}^{-n+m(k)-l+1}.\end{aligned}$$

The corresponding set $X_{k,\geqslant l}$ therefore has motivic measure

$$\mu(X_{k,\geqslant l}) = \mathbb{L}^{-\sigma(k)}\mu\bigl(\widetilde{X}_{k,\geqslant l}\bigr) = [\mathcal{X}_\tau]\mathbb{L}^{-n+m(k)-\sigma(k)-l+1}.$$

By additivity of the motivic measure, finally, we obtain that

$$\mu(X_{k,l}) = \mu(X_{k,\geqslant l}\setminus X_{k,\geqslant l+1}) = \mu(X_{k,\geqslant l})-\mu(X_{k,\geqslant l+1}) =$$
$$[\mathcal{X}_\tau]\mathbb{L}^{-n+m(k)-\sigma(k)-l+1}-[\mathcal{X}_\tau]\mathbb{L}^{-n+m(k)-\sigma(k)-l} = [\mathcal{X}_\tau](\mathbb{L}-1)\mathbb{L}^{-n+m(k)-\sigma(k)-l},$$

which concludes the proof of the lemma. ∎

Corollary 10.10. *Let f be as in Theorem 10.5 and suppose that τ is a compact face of Γ_f that is not contained in any coordinate hyperplane. Let $l\in\mathbf{Z}_{>0}$. Then $X_{\mathbf{Z}^n\cap\Delta_\tau,l}$ is a cylindric subset of $(t\mathbf{C}[[t]])^n$; i.e., $\mu(X_{\mathbf{Z}^n\cap\Delta_\tau,l})$ exists.*

Proof. Clearly, $X_{\mathbf{Z}^n\cap\Delta_\tau,l}$ equals the disjoint union

$$X_{\mathbf{Z}^n\cap\Delta_\tau,l} = \bigsqcup_{k\in\mathbf{Z}^n\cap\Delta_\tau} X_{k,l}. \tag{144}$$

We know that $X_{k,l}=\emptyset$ for $k\in\mathbf{Z}^n_{>0}$ with $m(k)>l$; hence we may restrict the above union to k satisfying $m(k)\leqslant l$. Choose $x\in\tau\cap\mathbf{R}^n_{>0}\neq\emptyset$. Then $m(k)=k\cdot x$ for all $k\in\Delta_\tau$. Moreover, $\{k\in\mathbf{R}^n_{\geqslant 0}\mid k\cdot x\leqslant l\}$ is a closed and bounded subset of $\mathbf{R}^n$, containing finitely many integral points. The union (144) so boils down to a finite disjoint union of sets $X_{k,l}$ that, by Lemma 10.9, are cylindric subsets of $(t\mathbf{C}[[t]])^n$. Consequently,

$$\mu(X_{\mathbf{Z}^n\cap\Delta_\tau,l}) = \sum_{\substack{k\in\mathbf{Z}^n\cap\Delta_\tau,\\ m(k)\leqslant l}}\mu(X_{k,l})$$

is well-defined. ∎

Lemma 10.11. *Let f be as in Theorem 10.5 and suppose that τ is a compact face of Γ_f that is contained in at least one coordinate hyperplane. Define $P_\tau\subseteq\{1,\ldots,n\}$ such that $\rho\in P_\tau$ if and only if $\tau\subseteq\{x_\rho=0\}$, and denote*

$$\mathcal{X}'_\tau = \Bigl\{(x_\rho)_{\rho\notin P_\tau}\in(\mathbf{C}^\times)^{n-|P_\tau|}\Bigm| f_\tau(x_\rho)_{\rho\notin P_\tau}=0\Bigr\}.$$

Let $k\in\mathbf{Z}^n\cap\Delta_\tau$, $\emptyset\subseteq P\subseteq P_\tau$, and put

$$k\vee P = k+\sum_{\rho\in P}\overline{\mathbf{Z}}_{\geqslant 0}e_\rho\subseteq\overline{\mathbf{Z}}^n_{>0}\cap\Delta_\tau^\infty.^{42}$$

[42] Hereby $(e_\rho)_{1\leqslant\rho\leqslant n}$ denotes the standard basis of $\mathbf{R}^n$, and $\overline{\mathbf{Z}}_{\geqslant 0}=\mathbf{Z}_{\geqslant 0}\cup\{\infty\}\subseteq\overline{\mathbf{R}}_{\geqslant 0}$.

Note that $m(k') = m(k) \in \mathbf{Z}_{>0}$ for all $k' \in k \vee P$. Finally, let $l \in \mathbf{Z}_{>0}$. Then

$$\mu(X_{k\vee P,l}) = \begin{cases} 0, & \text{if } l < m(k);\\ \big((\mathbb{L}-1)^{n-|P_\tau|} - [\mathcal{X}'_\tau]\big)(\mathbb{L}-1)^{|P_\tau|-|P|}\mathbb{L}^{-n+|P|-\sigma(k)}, & \text{if } l = m(k);\\ [\mathcal{X}'_\tau](\mathbb{L}-1)^{|P_\tau|-|P|+1}\mathbb{L}^{-n+|P|+m(k)-\sigma(k)-l}, & \text{if } l > m(k). \end{cases}$$

Proof. The proof is analogous to the proof of Lemma 10.9. Essential is that f_τ only depends on the variables x_ρ, $\rho \notin P_\tau$. The measure of

$$X_{k\vee P,\geqslant l} = \{\varphi \in (t\mathbf{C}[[t]])^n \mid \operatorname{ord}_t \varphi \in k \vee P \text{ and } \operatorname{ord}_t f(\varphi) \geqslant l\}$$

equals

$$\mu(X_{k\vee P,\geqslant l}) = [\mathcal{X}'_\tau](\mathbb{L}-1)^{|P_\tau|-|P|}\mathbb{L}^{-n+|P|+m(k)-\sigma(k)-l+1}$$

for $l > m(k)$. ∎

Corollary 10.12. *Let f be as in Theorem 10.5 and suppose that τ is a compact face of Γ_f that is contained in at least one coordinate hyperplane. Let $l \in \mathbf{Z}_{>0}$. Then $X_{\overline{\mathbf{Z}}^n_{>0}\cap\Delta^\infty_\tau,l}$ is a cylindric subset of $(t\mathbf{C}[[t]])^n$; i.e., $\mu(X_{\overline{\mathbf{Z}}^n_{>0}\cap\Delta^\infty_\tau,l})$ exists.*

Proof. Put $\emptyset \subsetneq P_\tau = \{\rho \mid \tau \subseteq \{x_\rho = 0\}\} \subsetneq \{1,\dots,n\}$ as usual, and suppose that Δ_τ is strictly positively spanned by the primitive vectors e_ρ, v_j; $\rho \in P_\tau, j \in J_\tau \neq \emptyset$; in $\mathbf{Z}^n_{\geqslant 0}\setminus\{0\}$. Choose a decomposition $\{\delta_i\}_{i\in I}$ of the cone Δ_τ into simplicial cones δ_i without introducing new rays, and assume that δ_i is strictly positively spanned by the linearly independent primitive vectors e_ρ, v_j; $\rho \in P_i, j \in J_i$; with $\emptyset \subseteq P_i \subseteq P_\tau$ and $\emptyset \subsetneq J_i \subseteq J_\tau$. Then the *extended simplicial cones*

$$\delta_i^\infty = \left\{\sum\nolimits_{\rho\in P_i} \bar\lambda_\rho e_\rho + \sum\nolimits_{j\in J_i} \lambda_j v_j \;\middle|\; \bar\lambda_\rho \in \overline{\mathbf{R}}_{>0}, \lambda_j \in \mathbf{R}_{>0} \text{ for all } \rho, j\right\}, \quad i \in I,$$

clearly partition Δ^∞_τ, and so we are looking at the finite disjoint union

$$X_{\overline{\mathbf{Z}}^n_{>0}\cap\Delta^\infty_\tau,l} = \bigsqcup_{i\in I} X_{\overline{\mathbf{Z}}^n_{>0}\cap\delta^\infty_i,l}.$$

Next we decompose $\overline{\mathbf{Z}}^n_{>0} \cap \delta_i^\infty$, and subsequently $X_{\overline{\mathbf{Z}}^n_{>0}\cap\delta^\infty_i,l}$, as

$$\overline{\mathbf{Z}}^n_{>0} \cap \delta_i^\infty = \bigsqcup_{k\in\mathbf{Z}^n\cap\delta'_i} k \vee P_i \quad \text{and} \quad X_{\overline{\mathbf{Z}}^n_{>0}\cap\delta^\infty_i,l} = \bigsqcup_{k\in\mathbf{Z}^n\cap\delta'_i} X_{k\vee P_i,l}, \tag{145}$$

with

$$\delta'_i = \left\{\sum\nolimits_{\rho\in P_i} h_\rho e_\rho + \sum\nolimits_{j\in J_i} \lambda_j v_j \;\middle|\; h_\rho \in (0,1], \lambda_j \in \mathbf{R}_{>0} \text{ for all } \rho, j\right\} \subseteq \delta_i.$$

Recall that $X_{k\vee P_i,l} = \emptyset$ for $k \in \mathbf{Z}^n_{>0}$ with $m(k) > l$. We may therefore restrict the second union of (145) to k satisfying $m(k) \leqslant l$. Choose $x = (x_1,\dots,x_n) \in \tau$ with $x_\rho > 0$ for all $\rho \notin P_\tau$. Then $m(k) = k\cdot x$ for all $k \in \delta'_i \subseteq \Delta_\tau$, and $\{k \in \delta'_i \mid k\cdot x \leqslant l\}$ is a bounded subset of $\mathbf{R}^n$, containing finitely many integral points. It follows that the second union of (145) is actually a finite disjoint union of cylindric[43] subsets $X_{k\vee P_i,l}$ of $(t\mathbf{C}[[t]])^n$. We conclude that the finite sum

$$\mu\big(X_{\overline{\mathbf{Z}}^n_{>0}\cap\Delta^\infty_\tau,l}\big) = \sum_{i\in I} \sum_{\substack{k\in\mathbf{Z}^n\cap\delta'_i,\\ m(k)\leqslant l}} \mu(X_{k\vee P_i,l})$$

[43]See Lemma 10.11.

is well-defined. ∎

Proof of Theorem 10.5. By definition we have[44]

$$Z_f^{\mathrm{mot},0}(T) = \mathbb{L}^{-n}\sum_{l\geqslant 1}[\mathcal{X}_l^0](\mathbb{L}^{-n}T)^l = \sum_{l\geqslant 1}\mu(X_l^0)T^l,$$

with

$$X_l^0 = \{\varphi \in (t\mathbf{C}[[t]])^n \mid \operatorname{ord}_t f(\varphi) = l\}, \qquad l \in \mathbf{Z}_{>0}.$$

If $\varphi \in X_l^0$, then $\operatorname{ord}_t \varphi \in \overline{\mathbf{Z}}_{>0}^n$ and $m(\operatorname{ord}_t \varphi) \leqslant \operatorname{ord}_t f(\varphi) = l < \infty$. Further, $\{\Delta_\tau^\infty \mid \tau \text{ is a compact face of } \Gamma_f\}$ forms a partition of $\{k \in \overline{\mathbf{R}}_{>0}^n \mid m(k) < \infty\}$. Hence we may write each X_l^0 as the finite disjoint union

$$X_l^0 = \bigsqcup_\tau X_{\overline{\mathbf{Z}}_{>0}^n\cap\Delta_\tau^\infty,l} = \bigsqcup_{\substack{\tau,\\ P_\tau=\emptyset}} X_{\mathbf{Z}^n\cap\Delta_\tau,l} \sqcup \bigsqcup_{\substack{\tau,\\ P_\tau\neq\emptyset}} X_{\overline{\mathbf{Z}}_{>0}^n\cap\Delta_\tau^\infty,l},$$

where all unions are over compact faces τ of Γ_f, and $P_\tau = \{\rho \mid \tau \subseteq \{x_\rho = 0\}\}$ as usual. By Corollaries 10.10 and 10.12, all $X_{\mathbf{Z}^n\cap\Delta_\tau,l}$ and $X_{\overline{\mathbf{Z}}_{>0}^n\cap\Delta_\tau^\infty,l}$ are cylindric subsets of $(t\mathbf{C}[[t]])^n$, which allows us to write $\mu(X_l^0)$ as the finite sum

$$\mu(X_l^0) = \sum_{\substack{\tau,\\ P_\tau=\emptyset}} \mu\bigl(X_{\mathbf{Z}^n\cap\Delta_\tau,l}\bigr) + \sum_{\substack{\tau,\\ P_\tau\neq\emptyset}} \mu\bigl(X_{\overline{\mathbf{Z}}_{>0}^n\cap\Delta_\tau^\infty,l}\bigr).$$

This leads to

$$Z_f^{\mathrm{mot},0}(T) = \sum_{\substack{\tau,\\ P_\tau=\emptyset}}\sum_{l\geqslant 1}\mu\bigl(X_{\mathbf{Z}^n\cap\Delta_\tau,l}\bigr)T^l + \sum_{\substack{\tau,\\ P_\tau\neq\emptyset}}\sum_{l\geqslant 1}\mu\bigl(X_{\overline{\mathbf{Z}}_{>0}^n\cap\Delta_\tau^\infty,l}\bigr)T^l.$$

If τ is not contained in any coordinate hyperplane, then by Corollary 10.10, we have

$$\sum_{l\geqslant 1}\mu\bigl(X_{\mathbf{Z}^n\cap\Delta_\tau,l}\bigr)T^l$$

$$= \sum_{l\geqslant 1}\sum_{\substack{k\in\mathbf{Z}^n\cap\Delta_\tau,\\ m(k)\leqslant l}} \mu(X_{k,l})T^l \tag{146}$$

$$= \sum_{k\in\mathbf{Z}^n\cap\Delta_\tau}\sum_{l\geqslant m(k)} \mu(X_{k,l})T^l \tag{147}$$

$$= \sum_{k\in\mathbf{Z}^n\cap\Delta_\tau} \mu\bigl(X_{k,m(k)}\bigr)T^{m(k)} + \sum_{k\in\mathbf{Z}^n\cap\Delta_\tau}\sum_{l\geqslant m(k)+1} \mu(X_{k,l})T^l.$$

Replacing the motivic measures $\mu(\cdot)$ by the expressions found in Lemma 10.9, we obtain

$$\begin{aligned}\sum_{l\geqslant 1}\mu\bigl(X_{\mathbf{Z}^n\cap\Delta_\tau,l}\bigr)T^l = \bigl((\mathbb{L}-1)^n - [\mathcal{X}_\tau]\bigr)\mathbb{L}^{-n}\sum_{k\in\mathbf{Z}^n\cap\Delta_\tau}\mathbb{L}^{-\sigma(k)}T^{m(k)}\\ + [\mathcal{X}_\tau](\mathbb{L}-1)\mathbb{L}^{-n}\sum_{k\in\mathbf{Z}^n\cap\Delta_\tau}\mathbb{L}^{-\sigma(k)}\sum_{l\geqslant m(k)+1}\mathbb{L}^{m(k)-l}T^l,\end{aligned}$$

[44] Note the difference between $\mathcal{X}_l^0$ (see Definition 10.1) and X_l^0.

and since

$$\sum_{l\geqslant m(k)+1}\mathbb{L}^{m(k)-l}T^l = T^{m(k)}\sum_{l\geqslant 1}\mathbb{L}^{-l}T^l = T^{m(k)}\frac{\mathbb{L}^{-1}T}{1-\mathbb{L}^{-1}T}\in\mathcal{M}_{\mathbf{C}}[[T]],$$

we eventually find

$$\begin{aligned}\sum_{l\geqslant 1}\mu\big(X_{\mathbf{Z}^n\cap\Delta_\tau,l}\big)T^l \\ &= \left(\big(1-\mathbb{L}^{-1}\big)^n - \mathbb{L}^{-n}[\mathcal{X}_\tau]\frac{1-T}{1-\mathbb{L}^{-1}T}\right)\sum_{k\in\mathbf{Z}^n\cap\Delta_\tau}\mathbb{L}^{-\sigma(k)}T^{m(k)}.\end{aligned}$$

This last sum, denoted $S(\Delta_\tau)$, can be calculated as follows. First choose a decomposition $\{\delta_i\}_{i\in I}$ of the cone Δ_τ into simplicial cones δ_i without introducing new rays. Then $S(\Delta_\tau)=\sum_{i\in I}S(\delta_i)$, whereby

$$S(\delta_i)=\sum_{k\in\mathbf{Z}^n\cap\delta_i}\mathbb{L}^{-\sigma(k)}T^{m(k)}$$

for all $i\in I$. Next assume that the cone δ_i is strictly positively spanned by the linearly independent primitive vectors v_j, $j\in J_i$, in $\mathbf{Z}^n_{\geqslant 0}\setminus\{0\}$. Then $\mathbf{Z}^n\cap\delta_i$ equals the finite disjoint union $\bigsqcup_h h+\sum_{j\in J_i}\mathbf{Z}_{\geqslant 0}v_j$, where h runs through the elements of

$$\tilde{H}(v_j)_{j\in J_i}=\mathbf{Z}^n\cap\tilde{\Diamond}(v_j)_{j\in J_i}=\mathbf{Z}^n\cap\left\{\sum\nolimits_{j\in J_i}h_jv_j\;\middle|\;h_j\in(0,1]\text{ for all }j\in J_i\right\}.$$

Consequently,

$$S(\delta_i)=\sum_{h\in\tilde{H}(v_j)_j}\;\sum_{(\lambda_j)_j\in\mathbf{Z}^{|J_i|}_{\geqslant 0}}\mathbb{L}^{-\sigma\left(h+\sum_j\lambda_jv_j\right)}T^{m\left(h+\sum_j\lambda_jv_j\right)};$$

then exploiting the linearity[45] of $m(\cdot)$ on $\overline{\Delta_\tau}\supseteq\delta_i$, we find

$$\begin{aligned}S(\delta_i)&=\sum_{h\in\tilde{H}(v_j)_j}\mathbb{L}^{-\sigma(h)}T^{m(h)}\prod_{j\in J_i}\sum_{\lambda_j\geqslant 0}\left(\mathbb{L}^{-\sigma(v_j)}T^{m(v_j)}\right)^{\lambda_j}\\ &=\frac{\sum_{h\in\tilde{H}(v_j)_j}\mathbb{L}^{-\sigma(h)}T^{m(h)}}{\prod_{j\in J_i}\big(1-\mathbb{L}^{-\sigma(v_j)}T^{m(v_j)}\big)}\in\mathcal{M}_{\mathbf{C}}[[T]].\end{aligned}$$

Note that since all $m(v_j)$ are positive, we indeed obtain an element of $\mathcal{M}_{\mathbf{C}}[[T]]$.

The eventual formula for $\sum_{l\geqslant 1}\mu\big(X_{\mathbf{Z}^n\cap\Delta_\tau,l}\big)T^l$ is thus

$$\left(\big(1-\mathbb{L}^{-1}\big)^n-\mathbb{L}^{-n}[\mathcal{X}_\tau]\frac{1-T}{1-\mathbb{L}^{-1}T}\right)\sum_{i\in I}\frac{\sum_{h\in\tilde{H}(v_j)_{j\in J_i}}\mathbb{L}^{-\sigma(h)}T^{m(h)}}{\prod_{j\in J_i}\big(1-\mathbb{L}^{-\sigma(v_j)}T^{m(v_j)}\big)}\in\mathcal{M}_{\mathbf{C}}[[T]],$$

as announced in the theorem. To rigorously prove that (146) equals this last expression in $\mathcal{M}_{\mathbf{C}}[[T]]$, in particular to explain the change of summation order in going from (146) to (147), one compares the coefficients of T^l in both elements and finds twice the same finite sum in $\mathcal{M}_{\mathbf{C}}$.

From now suppose that τ is contained in at least one coordinate hyperplane; i.e., $P_\tau\neq\emptyset$. Let Δ_τ be strictly positively spanned by the primitive vectors e_ρ, v_j; $\rho\in P_\tau, j\in J_\tau\neq\emptyset$; in $\mathbf{Z}^n_{\geqslant 0}\setminus\{0\}$. Choose a decomposition $\{\delta_i\}_{i\in I}$ of Δ_τ into simplicial cones δ_i without introducing new rays, and assume that δ_i is strictly positively

[45]Recall that for any $x\in\tau$ we have that $m(k)=k\cdot x$ for all $k\in\overline{\Delta_\tau}$.

spanned by the linearly independent primitive vectors e_ρ, v_j; $\rho \in P_i, j \in J_i$; with $\emptyset \subseteq P_i \subseteq P_\tau$ and $\emptyset \subsetneq J_i \subseteq J_\tau$. Finally, put $\delta_i' = \tilde{\Diamond}(e_\rho)_{\rho\in P_i} + \mathrm{cone}(v_j)_{j\in J_i}$ as before.

We proceed as in the $P_\tau = \emptyset$ case. Corollary 10.12 yields

$$\begin{aligned}
&\sum_{l\geqslant 1} \mu\big(X_{\bar{\mathbf{Z}}^n_{>0}\cap\Delta_\tau^\infty,l}\big)T^l \\
&\qquad = \sum_{l\geqslant 1}\sum_{i\in I}\sum_{\substack{k\in\mathbf{Z}^n\cap\delta_i',\\ m(k)\leqslant l}} \mu(X_{k\vee P_i,l})T^l \in \mathcal{M}_{\mathbf{C}}[[T]] \\
&\qquad = \sum_{i\in I}\sum_{k\in\mathbf{Z}^n\cap\delta_i'} \mu\big(X_{k\vee P_i,m(k)}\big)T^{m(k)} + \sum_{i\in I}\sum_{k\in\mathbf{Z}^n\cap\delta_i'}\sum_{l\geqslant m(k)+1} \mu(X_{k\vee P_i,l})T^l.
\end{aligned}$$

Then applying Lemma 10.11, we find

$$\begin{aligned}
&\sum_{l\geqslant 1} \mu\big(X_{\bar{\mathbf{Z}}^n_{>0}\cap\Delta_\tau^\infty,l}\big)T^l \\
&= \big((\mathbb{L}-1)^{n-|P_\tau|} - [\mathcal{X}_\tau']\big)\mathbb{L}^{-n}\sum_{i\in I}(\mathbb{L}-1)^{|P_\tau|-|P_i|}\mathbb{L}^{|P_i|}\sum_{k\in\mathbf{Z}^n\cap\delta_i'} \mathbb{L}^{-\sigma(k)}T^{m(k)} \\
&\qquad + [\mathcal{X}_\tau']\mathbb{L}^{-n}\sum_{i\in I}(\mathbb{L}-1)^{|P_\tau|-|P_i|+1}\mathbb{L}^{|P_i|}\sum_{k\in\mathbf{Z}^n\cap\delta_i'}\mathbb{L}^{-\sigma(k)}\sum_{l\geqslant m(k)+1}\mathbb{L}^{m(k)-l}T^l \\
&= \left(\big(1-\mathbb{L}^{-1}\big)^{n-|P_\tau|} - \mathbb{L}^{-(n-|P_\tau|)}[\mathcal{X}_\tau']\frac{1-T}{1-\mathbb{L}^{-1}T}\right) \\
&\qquad\qquad\qquad \cdot \sum_{i\in I}\big(1-\mathbb{L}^{-1}\big)^{|P_\tau|-|P_i|}\sum_{k\in\mathbf{Z}^n\cap\delta_i'}\mathbb{L}^{-\sigma(k)}T^{m(k)}.
\end{aligned}$$

This last double sum, which we denote by $S(\Delta_\tau)'$, can be calculated in the same way as we calculated $S(\Delta_\tau)$ in the $P_\tau = \emptyset$ case. We obtain

$$S(\Delta_\tau)' = \sum_{i\in I}\big(1-\mathbb{L}^{-1}\big)^{|P_\tau|-|P_i|}\frac{\sum_h \mathbb{L}^{-\sigma(h)}T^{m(h)}}{\prod_{j\in J_i}\big(1-\mathbb{L}^{-\sigma(v_j)}T^{m(v_j)}\big)} \in \mathcal{M}_{\mathbf{C}}[[T]],$$

where h runs through the elements of the set

$$\tilde{H}(e_\rho, v_j)_{\rho,j} = \mathbf{Z}^n \cap \left\{\sum\nolimits_{\rho\in P_i} h_\rho e_\rho + \sum\nolimits_{j\in J_i} h_j v_j \;\middle|\; h_\rho, h_j \in (0,1] \text{ for all } \rho, j\right\}.$$

Note again that since all $m(v_j)$ are positive, we indeed find an element of $\mathcal{M}_{\mathbf{C}}[[T]]$. This concludes the proof of Theorem 10.5. ■

10.3. **A proof of the main theorem in the motivic setting.** In this final subsection we explain why (and how) Theorem 10.3 can be proved in the same way as Theorem 0.12. Let us start with a small overview.

Let f be as in Theorem 10.3. By the general rationality result of Denef–Loeser, we know that there exists a finite set $\widetilde{S} \subseteq \mathbf{Z}^2_{>0}$ such that

$$Z_f^{\mathrm{mot},0}(T) \in \mathcal{M}_{\mathbf{C}}\left[\frac{\mathbb{L}^{-\sigma}T^m}{1-\mathbb{L}^{-\sigma}T^m}\right]_{(m,\sigma)\in\widetilde{S}} \subseteq \mathcal{M}_{\mathbf{C}}[[T]].$$

Our formula for non-degenerate f (Theorem 10.5), on the other hand, yields

$$Z_f^{\mathrm{mot},0}(T) \in \mathcal{M}_{\mathbf{C}}[T]\left[\frac{1}{1-\mathbb{L}^{-\sigma}T^m}\right]_{(m,\sigma)\in S} \subseteq \mathcal{M}_{\mathbf{C}}[[T]],$$

whereby

(148) $S = \{(1,1)\} \cup \{(m(v), \sigma(v)) \mid v$ is the primitive vector associated to a facet of Γ_f that is not contained in any coordinate hyperplane$\} \subseteq \mathbf{Z}^2_{>0}$.

Now we want to prove that there exists a subset $S' \subseteq S$ such that

$$Z_f^{\mathrm{mot},0}(T) \in \mathcal{M}_{\mathbf{C}}[T]\left[\frac{1}{1-\mathbb{L}^{-\sigma}T^m}\right]_{(m,\sigma)\in S'}, \tag{149}$$

and such that $e^{-2\pi i\sigma/m}$ is an eigenvalue of monodromy (in the sense of the theorem) for each $(m,\sigma) \in S'$.

Let us introduce some notations and terminology. Consider the set S from (148), and put $Q = \{\sigma/m \mid (m,\sigma) \in S\} \subseteq \mathbf{Q}_{>0}$. Let $q \in Q$ and let τ be a facet of Γ_f that is not contained in any coordinate hyperplane. We say that τ contributes to q if $\sigma(v)/m(v) = q$, with v the unique primitive vector in $\mathbf{Z}^n_{\geqslant 0} \setminus \{0\}$ perpendicular to τ. We shall call a ratio $q \in Q$ *good* if

- $q = 1$,
- or q is contributed by a facet of Γ_f that is not a B_1-facet,
- or q is contributed by two B_1-facets of Γ_f that are <u>not</u> B_1 for a same variable and that have an edge in common.

We shall call $q \in Q$ *bad* if q is not good, i.e., if

- $q \neq 1$;
- and q is only contributed by B_1-facets of Γ_f;
- and for any pair of contributing B_1-facets, we have that
  - either they are B_1-facets for a same variable,
  - or they have at most one point in common.

Finally, we shall call a facet τ of Γ_f *bad* if it contributes to a bad $q \in Q$. This implies that τ is a B_1-facet.

Let us now define

$$S' = \{(m,\sigma) \in S \mid \sigma/m \text{ is good}\} \subseteq S.$$

Then by Theorem 0.34 and Proposition 0.39 by Lemahieu and Van Proeyen, we know that $e^{-2\pi i\sigma/m}$ is an eigenvalue of monodromy for each $(m,\sigma) \in S'$. It remains to prove that (149) holds for the S' proposed above.

The formula for $Z_f^{\mathrm{mot},0}(T)$ in Theorem 10.5 associates a term to every compact face τ of Γ_f. If τ is not contained in any bad facet, then its associated term clearly belongs to

$$\mathcal{M}_{\mathbf{C}}[T]\left[\frac{1}{1-\mathbb{L}^{-\sigma}T^m}\right]_{(m,\sigma)\in S'}.$$

Hence it suffices to consider the sum of the terms associated to bad B_1-simplices or compact subfaces of bad B_1-facets. We will refer to this sum as the *relevant* part of $Z_f^{\mathrm{mot},0}(T)$.

If we look at the formula carefully, we see that it is a rational expression (with integer coefficients) in $\mathbb{L}$ and T, except for the presence of $[\mathcal{X}_\tau]$ and $[\mathcal{X}'_\tau]$ in L_τ and L'_τ, respectively. Fortunately, for the *relevant* faces, these classes have a fairly simple form. For any vertex V, we have $[\mathcal{X}_V] = [\mathcal{X}'_V] = 0$. If $[CD]$ is any edge with one vertex in a coordinate hyperplane and the other vertex at distance one of this hyperplane, then $[\mathcal{X}_{[CD]}] = (\mathbb{L}-1)^2$ if $[CD]$ is not contained in any coordinate hyperplane, and $[\mathcal{X}'_{[CD]}] = \mathbb{L}-1$ otherwise.

Lastly, let τ_0 be a B_1-simplex with a base[46] $[AB]$. Then we have the relation

$$[\mathcal{X}_{\tau_0}] = (\mathbb{L}-1)^2 - [\mathcal{X}'_{[AB]}]. \tag{150}$$

Let us write down the contributions of τ_0 and $[AB]$ to $Z_f^{\mathrm{mot},0}(T)$. If we denote by v_0 the unique primitive vector in $\mathbf{Z}^n_{\geqslant 0} \setminus \{0\}$ perpendicular to τ_0, then

$$\begin{aligned}
&L_{\tau_0} S(\Delta_{\tau_0}) + L'_{[AB]} S(\Delta_{[AB]})' \\
&= \left[\left(1-\mathbb{L}^{-1}\right)^3 - \mathbb{L}^{-3}\left((\mathbb{L}-1)^2 - [\mathcal{X}'_{[AB]}]\right)\frac{1-T}{1-\mathbb{L}^{-1}T}\right] \frac{\mathbb{L}^{-\sigma(v_0)}T^{m(v_0)}}{1-\mathbb{L}^{-\sigma(v_0)}T^{m(v_0)}} \\
&\qquad + \left[\left(1-\mathbb{L}^{-1}\right)^2 - \mathbb{L}^{-2}[\mathcal{X}'_{[AB]}]\frac{1-T}{1-\mathbb{L}^{-1}T}\right] \frac{\mathbb{L}^{-\sigma(v_0)-1}T^{m(v_0)}}{1-\mathbb{L}^{-\sigma(v_0)}T^{m(v_0)}} \\
&= \frac{\left(1-\mathbb{L}^{-1}\right)^3}{1-\mathbb{L}^{-1}T} \frac{\mathbb{L}^{-\sigma(v_0)}T^{m(v_0)}}{1-\mathbb{L}^{-\sigma(v_0)}T^{m(v_0)}}. \qquad (151)
\end{aligned}$$

As we observed in the p-adic case, Identity (150), together with the fact that $\operatorname{mult}\Delta_{[AB]} = 1$, causes the cancellation of $[\mathcal{X}'_{[AB]}]$. We shall call (151) the contribution of τ_0 and $[AB]$ to $Z_f^{\mathrm{mot},0}(T)$ after cancellation.

After these cancellations (one for every bad B_1-simplex), the relevant part of $Z_f^{\mathrm{mot},0}(T)$ is indeed a rational expression in $\mathbb{L}$ and T. More precisely, it is an element of the ring

$$\mathbf{Z}[\mathbb{L},\mathbb{L}^{-1}][T]\left[\frac{1}{1-\mathbb{L}^{-\sigma}T^m}\right]_{(m,\sigma)\in S} \subseteq \mathcal{M}_{\mathbf{C}}[T]\left[\frac{1}{1-\mathbb{L}^{-\sigma}T^m}\right]_{(m,\sigma)\in S}, \tag{152}$$

whereby $\mathbf{Z}[\mathbb{L},\mathbb{L}^{-1}] \subseteq \mathcal{M}_{\mathbf{C}}$ denotes the smallest subring of $\mathcal{M}_{\mathbf{C}}$ containing $\mathbf{Z}, \mathbb{L}$, and $\mathbb{L}^{-1}$. We can now replace $\mathbb{L}$ by a new indeterminate U and study the relevant part of $Z_f^{\mathrm{mot},0}(T)$ in the ring

$$\mathbf{Z}[U,U^{-1}][T]\left[\frac{1}{1-U^{-\sigma}T^m}\right]_{(m,\sigma)\in S} \subseteq \mathbf{Z}[U,U^{-1}](T), \tag{153}$$

where $\mathbf{Z}[U,U^{-1}]$ is the ring of formal Laurent polynomials over $\mathbf{Z}$. The advantage is that the coefficients of T now live in the unique factorization domain $\mathbf{Z}[U,U^{-1}]$. There clearly exists a surjective ring morphism from (153) to (152); so if we can prove equality in (153), equality in (152) follows.

The goal is now to prove that the relevant part of $Z_f^{\mathrm{mot},0}(T)$ (seen as an element in this new ring) also belongs to

$$\mathbf{Z}[U,U^{-1}][T]\left[\frac{1}{1-U^{-\sigma}T^m}\right]_{(m,\sigma)\in S'}.$$

The advantage of working in a unique factorization domain is that we may now choose a bad $q \in Q$ randomly and restrict ourselves to proving that the relevant

[46]If a B_1-simplex τ_0 has two vertices A and B in a coordinate hyperplane and one vertex at distance one of this hyperplane, then we shall call $[AB]$ a base of τ_0. A B_1-simplex has by definition at least one base, but can have several.

part of $Z_f^{\mathrm{mot},0}(T)$ is an element of

$$\mathbf{Z}[U,U^{-1}][T]\left[\frac{1}{1-U^{-\sigma}T^m}\right]_{(m,\sigma)\in S\setminus S_q},$$

with $S_q = \{(m,\sigma)\in S \mid \sigma/m = q\} \subseteq S$ and $S' \subseteq S\setminus S_q \subseteq S$. Indeed, if $\sigma_1/m_1 \neq \sigma_2/m_2$, then $1-U^{-\sigma_1}T^{m_1}$ and $1-U^{-\sigma_2}T^{m_2}$ have no common irreducible factors in $\mathbf{Z}[U,U^{-1}][T]$.

So from now on, q is a fixed bad ratio in Q. We define a *q-cluster* as a family $\mathcal{C}$ of (bad B_1-) facets contributing to q, such that for any two facets $\tau,\tau'\in\mathcal{C}$, there exists a chain $\tau=\tau_0,\tau_1,\ldots,\tau_t=\tau'$ of B_1-facets in $\mathcal{C}$ with the property that τ_{j-1} and τ_j share an edge for all $j\in\{1,\ldots,t\}$. A *maximal q-cluster* is a q-cluster that is not contained in a strictly bigger one. Note that every facet contributing to q is contained in precisely one maximal q-cluster. Also note that the supports[47] of two distinct maximal q-clusters may share a vertex of Γ_f, but never share an edge.

Let V be a vertex of Γ_f, and let τ_j, $j\in J$, be all the facets of Γ_f that contain V. Denote for each $j\in J$, by v_j the unique primitive vectors in $\mathbf{Z}^n_{\geqslant 0}\setminus\{0\}$ perpendicular to τ_j. Then Δ_V is the cone strictly positively spanned by the vectors v_j, $j\in J$. Let $\{\delta_i\}_{i\in I}$ be a decomposition of Δ_V into simplicial cones δ_i without introducing new rays, and assume that δ_i is strictly positively spanned by the vectors v_j, $j\in J_i$. We shall say that a cone δ_i *meets* a q-cluster $\mathcal{C}$ if $\{\tau_j \mid j\in J_i\}\cap\mathcal{C}\neq\emptyset$. We shall call $\{\delta_i\}_{i\in I}$ a *nice* decomposition if every δ_i meets at most one maximal q-cluster. By construction of the maximal q-clusters, a nice decomposition of Δ_V always exists.

Let us now choose a nice decomposition $\{\delta_{V,i}\}_{i\in I_V}$ of Δ_V for every relevant vertex V of Γ_f. The relevant part of $Z_f^{\mathrm{mot},0}(T)$ contains a term for every such V. According to the formula in Theorem 10.5, this term can be split up into terms, one for each simplicial cone $\delta_{V,i}$ in the decomposition of Δ_V. Let $\mathcal{C}$ be a maximal q-cluster. We define the part of $Z_f^{\mathrm{mot},0}(T)$ associated to $\mathcal{C}$ as the sum of the following terms:

- for each B_1-simplex $\tau\in\mathcal{C}$ with chosen base b_τ, the contribution of τ and b_τ to $Z_f^{\mathrm{mot},0}(T)$ after cancellation;
- the terms associated to the other compact edges of the B_1-facets in $\mathcal{C}$;
- the terms associated to the simplicial cones $\delta_{V,i}$ that meet $\mathcal{C}$.

Note that in this way no term is assigned to more than one maximal q-cluster.

It follows that the relevant part of $Z_f^{\mathrm{mot},0}(T)$ is given by

$$\sum_{\substack{\mathcal{C}\text{ maximal}\\ q\text{-cluster}}}(\text{part of } Z_f^{\mathrm{mot},0}(T)\text{ associated to }\mathcal{C}) + (\text{sum of remaining terms}).$$

By construction the sum of the remaining terms is certainly an element of

$$\mathbf{Z}[U,U^{-1}][T]\left[\frac{1}{1-U^{-\sigma}T^m}\right]_{(m,\sigma)\in S\setminus S_q}. \tag{154}$$

The problem is therefore reduced to proving that for every maximal q-cluster $\mathcal{C}$, the part of $Z_f^{\mathrm{mot},0}(T)$ associated to $\mathcal{C}$ belongs to (154).

A maximal q-cluster contains no more than two B_1-facets, otherwise q would equal one (see Case VI in Section 7). Moreover, two B_1-facets belonging to the same maximal q-cluster, are always B_1 for a same variable, otherwise q would be

[47]By the support of a q-cluster we mean the union of its facets.

good. This leaves us five possible configurations of a maximal q-cluster $\mathcal{C}$; it consists of

(i) one B_1-simplex τ_0,
(ii) or one non-compact B_1-facet τ_0,
(iii) or two B_1-simplices τ_0 and τ_1 for a same variable,
(iv) or two non-compact B_1-facets τ_0 and τ_1 for a same variable,
(v) or one non-compact B_1-facet τ_0 and one B_1-simplex τ_1 for a same variable.

Pictures can be found in Figures 4, 5, 6, 8, and 9, respectively.

In Cases (i) and (ii), the part of $Z_f^{\mathrm{mot},0}(T)$ associated to $\mathcal{C}$ has the form

$$\frac{N_1(T)}{(1-U^{-1}T)F_0F_1F_2} \in \mathbf{Z}[U,U^{-1}][T]\left[\frac{1}{1-U^{-\sigma}T^m}\right]_{(m,\sigma)\in S} \subseteq \mathbf{Z}[U,U^{-1}](T),$$

while in Cases (iii)–(v), it has the form

$$\frac{N_2(T)}{(1-U^{-1}T)F_0F_1F_2F_3} \in \mathbf{Z}[U,U^{-1}][T]\left[\frac{1}{1-U^{-\sigma}T^m}\right]_{(m,\sigma)\in S} \subseteq \mathbf{Z}[U,U^{-1}](T).$$

Hereby $N_1(T)$ and $N_2(T)$ are polynomials in T with coefficients in $\mathbf{Z}[U,U^{-1}]$, and so are

$$F_j = 1-U^{-\sigma_j}T^{m_j} \subseteq \mathbf{Z}[U,U^{-1}][T]; \qquad j=0,\dots,3.$$

In Cases (i) and (ii), the factor F_0 corresponds to τ_0, while F_1 and F_2 correspond to *neighbor facets*[48] of τ_0. It follows that $\sigma_0/m_0 = q$ and $\sigma_j/m_j \neq q$ for $j=1,2$. In Cases (iii)–(v), factors F_0 and F_1 correspond to τ_0 and τ_1, where F_2 and F_3 come from neighbor facets[48] of τ_0 and τ_1. We have $\sigma_0/m_0 = \sigma_1/m_1 = q$ and $\sigma_j/m_j \neq q$ for $j=2,3$.

Finally everything boils down to proving that (depending on the case)

$$F_0 \mid N_1(T) \qquad \text{or} \qquad F_0F_1 \mid N_2(T) \tag{155}$$

in the polynomial ring $\mathbf{Z}[U,U^{-1}][T]$. As F_0 and F_1 are monic polynomials (in the sense that their leading coefficients are units of $\mathbf{Z}[U,U^{-1}]$), the divisibility conditions (155) can be investigated equivalently over the fraction field $\mathbf{Q}(U)$ of $\mathbf{Z}[U,U^{-1}]$. Now we can decide divisibility by looking at the roots of F_0 and F_1 in some algebraic closure of the coefficient field $\mathbf{Q}(U)$. We shall consider the field $\overline{\mathbf{Q}}\{\{U\}\}$ of formal Puiseux series over the field $\overline{\mathbf{Q}}$ of algebraic numbers.

The polynomial $F_j = 1-U^{-\sigma_j}T^{m_j}$ has m_j distinct roots

$$T_k^{(j)} = U^{\frac{\sigma_j}{m_j}} e^{\frac{2k\pi i}{m_j}}; \qquad k=0,1,\dots,m_j-1;$$

in $\overline{\mathbf{Q}}\{\{U\}\}$ for $j=0,1$. Hence F_0 divides $N_1(T)$ if and only if $N_1\big(T_k^{(0)}\big)=0$ in $\overline{\mathbf{Q}}\{\{U\}\}$ for all k. In Cases (iii)–(v), we may conclude that $F_0F_1 \mid N_2(T)$ as soon as $N_2\big(T_k^{(j)}\big)=0$ for all k and $j=0,1$ *and* $N_2'(T)$ vanishes in all common roots

$$T_k^{(0,1)} = U^q e^{\frac{2k\pi i}{\gcd(m_0,m_1)}}; \qquad k=0,1,\dots,\gcd(m_0,m_1)-1;$$

of F_0 and F_1 in $\overline{\mathbf{Q}}\{\{U\}\}$.

The proof of each of the identities $N_1\big(T_k^{(0)}\big)=0$, $N_2\big(T_k^{(j)}\big)=0$, $N_2'\big(T_k^{(0,1)}\big)=0$ is identical to one of the 'residue vanishing proofs' in Cases I–V in Sections 2–6. For example, in Case (iii) of the current proof, the proof of $N_2\big(T_k^{(j)}\big)=0$ for a simple

[48]By a neighbor facet we mean a facet sharing an edge. A factor will appear in the denominator for every neighbor facet that does not lie in a coordinate hyperplane.

root $T_k^{(j)}$ of F_0F_1, corresponds to the proof of $R_1 = 0$ in Case I. For a double root $T_k^{(0,1)}$ of F_0F_1, the proofs of $N_2\big(T_k^{(0,1)}\big) = 0$ and $N_2'\big(T_k^{(0,1)}\big) = 0$ are completely analogous to the proofs of $R_2 = 0$ and $R_1 = 0$, respectively, in Case III. This ends the sketch of the proof of the main theorem in the motivic setting.

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